# Problem Set 1 Solutions

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## 1 Problem 1

1. $A \in P(A)$: Always true, because of the definition of the power set. $A \subseteq A \Rightarrow A \in P(A)$
   - Example: $\{a,b\} \in \{\{a\},\{b\},\{a,b\},\emptyset\}$
2. $A \subseteq P(A)$: Sometimes true. It can be shown that powersets of powersets of empty sets have the property that they are subsets of their own powersets. Otherwise, it is not necessarily true.
   - Example 1 : $A \subseteq P(A)$

$$\begin{aligned} \emptyset &\subseteq \{\emptyset\} \\ \{\emptyset\} &\subseteq \{\emptyset,\{\emptyset\}\} \\ \{\emptyset,\{\emptyset\}\} &\subseteq \Big\{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\Big\} \end{aligned}$$

   - Example 2 : $A \not\subseteq P(A)$

$$\{a\} \quad \not\subseteq \quad \{\emptyset,\{a\}\}$$

3. $(|A| \leq |B|) \Rightarrow (A \subseteq B)$: Sometimes true. The cardinality of a set says nothing about its elements. The sets $A$ and $B$ might even be disjoint.
   - Example 1: $|\{1,2\}| \leq |\{1,2,3\}|$ and $\{1,2\} \subseteq \{1,2,3\}$
   - Example 2: $|\{1,2\}| \leq |\{red, green, blue\}|$ and $\{1,2\} \not\subseteq \{red, green, blue\}$
4. $(A \subseteq B) \Rightarrow (|A| \leq |B|)$: Always true, because $|A| > |B| \Rightarrow$ There is $x \in A$ such that $x \notin B \Leftrightarrow A \not\subseteq B$.
   - Example : $\{1,2\} \ \subseteq \{1,2,3\} \Rightarrow |\{1,2\}| \leq |\{1,2,3\}|$

## 2 Problem 2

1. $A \in B$, $A \subseteq B$, and $P(A) \subseteq B$.
   - $A = \emptyset, B = \{\emptyset\}$
   - $P(A) = \{\emptyset\} \subseteq B$
   - These sets are as small as possible, since $A$ has no elements at all, and $A \in B \Rightarrow |B| \geq 1$; and we have $|B| = 1$.
2. $(\mathbb{N} \cap A) \in A$, $B \subset A$, and $P(B) \subseteq A$.
   - $A = \{\emptyset\}, B = \emptyset$
   - $(\mathbb{N} \cap A) = \emptyset \in A$

- $B = \emptyset \subset A = \{\emptyset\}$
- $P(B) = \{\emptyset\} \subseteq A = \{\emptyset\}$
- These sets are as small as possible since $B$ has no elements and $(\mathbb{N} \cap A) \in A \Rightarrow |A| \geq 1$, and we have $|A| = 1$.

3. $A \subseteq (P(P(B)) - P(A))$.
   - $A = \emptyset, B = \emptyset$
   - An empty set is a subset of every set.
4. $A \supseteq (P(P(B)) - P(A))$.
   - $A = \{\emptyset\}, B = \emptyset$
   - $A = P(B)$, so $P(A) = P(P(B))$ and $(P(P(B)) - P(A)) = \emptyset$, a subset of every set
   - These sets are as small as possible, since at least one of the sets must be nonempty. Proof by contradiction:
     Let A and B be empty sets ($|A| = |B| = \emptyset$). Then
     $|P(A)| = 1$, $|P(B)| = 1$, and $|P(P(B))| = 2$
     $\Rightarrow |P(P(B)) - P(A)| \geq 1$
     $\Rightarrow |A| \geq 1$, a contradiction, since $A$ is empty.

# 3 Problem 3

1. $T(S(R)) \not\subseteq S(T(R))$. Proof by counterexample (See Figure 1):
   Let
   $$R = \{(a,b),(a,c)\}$$
   Then
   $$\begin{aligned} T(R) &= \{(a,b),(a,c)\} \\ S(R) &= \{(a,b),(a,c),(c,a),(b,a)\} \\ T(S(R)) &= \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\} \\ S(T(R)) &= \{(a,b),(b,a),(a,c),(c,a)\} \end{aligned}$$

Therefore,

$$T(S(R)) \not\subseteq S(T(R))$$

2. $S(T(R)) \subseteq T(S(R))$
   *Proof:*A Sketch of Yoni's Proof

   Let $R$ be a binary relation on a set $S$.

   *Lemma* 1: $T(R)$ and $S(R)$ exist and are unique. One can construct $T(R)$ by adding elements to $S(R)$, one at a time, so that $(a,b)$ is added only if $(a,c)$ and $(c,b)$ are already in the relation, until all possibilities are exhausted.

   *Lemma* 2: $(A \subseteq B) \Rightarrow (T(A) \subseteq T(B))$.

   *Lemma* 3: $(A \subseteq B) \Rightarrow (S(A) \subseteq S(B))$.

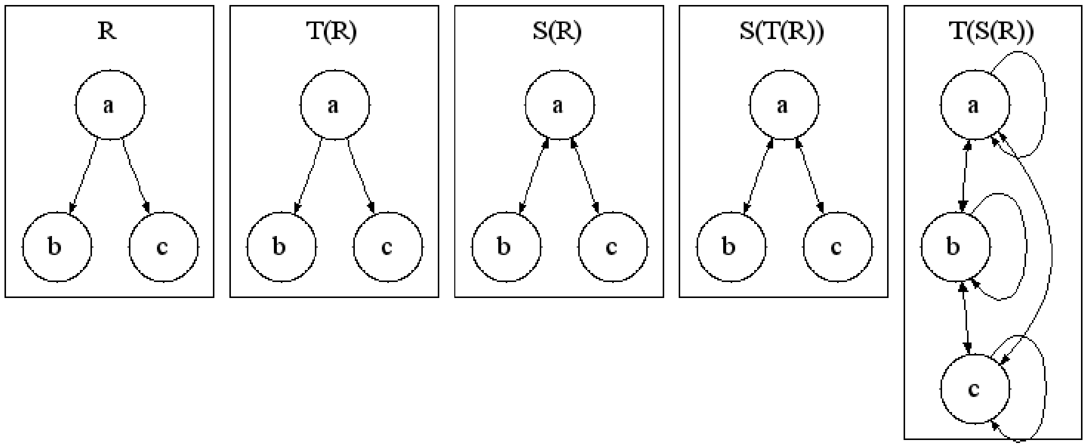


Figure 1: $T(S(R)) \not\subseteq S(T(R))$

These lemmas are left without proof since we didn't prove *Lemma* 1.

*Lemma* 4: $T(S(R)$ is symmetric.

*Proof:*: Proof by invariant. Let us obtain $T(S(R))$ from $S(R)$ by adding elements to $S(R)$ one at a time, as in *Lemma* 1

The claim is that each time one adds an element $(a, b)$ that makes the relation non-symmetric, one must also add $(b, a)$ to obtain a transitive set in the end. According to our construction, we only add $(a, b)$ whenever $(a, c)$ and $(c, b)$ are in the relation. If we are adding $(a, b)$ to a symmetric relation, then $(c, a)$ and $(b, c)$ must be in that relation as well. Since we are to obtain a transitive relation in the end, we must add $(b, a)$ as well, preserving symmetry.

*Proof of the theorem:*

(a) $S(T(S(R))) = T(S(R))$ (from *Lemma* 4);

(b) $R \subseteq S(R) \Rightarrow T(R) \subseteq T(S(R))$ (from *Lemma* 2)

(c) $T(R(\subseteq T(S(R)) \Rightarrow S(T(R)) \subseteq S(T(S(R)))$ (from *Lemma* 3)

(d) But $S(T(S(R))) = T(S(R))$, so $S(T(R)) \subseteq T(S(R))$, as desired.

# 4 Problem 4

Show that each function $f : \mathbb{N} \rightarrow \mathbb{N}$ has the listed properties.

1. $f(x) = 2x$ (one-to-one but not onto)
   - $f$ is one-one, since $(x \neq y) \Rightarrow (2x \neq 2y) \Rightarrow (f(x) \neq f(y))$
   - $f$ is not onto, since there is no integer $x$ such that $f(x) = 1$ (in fact, for all integers $k$, there is no integer $x$ such that $f(x) = 2k + 1$).
2. $f(x) = x + 1$ (one-to-one but not onto)
   - $f$ is one-one since $(x \neq y) \Rightarrow (x + 1 \neq y + 1) \Rightarrow (f(x) \neq f(y))$

- $f$ is not onto since there is no natural number $n$ such that $f(n) = (n+1) = 0$.

3. $f(x) =$ if $x$ is odd then $x - 1$ else $x + 1$ (bijective)

   - $f$ is one-one:
     *Proof:*Let $f(x) = f(y)$ for some $x, y \in \mathbb{N}$. We shall show that this implies $x = y$. There are three possible cases:
     (a) $x$ and $y$ are both odd. Then $f(x) = x - 1$ and $f(y) = y - 1$.
     Therefore, $(f(x) = f(y)) \Rightarrow (x - 1 = y - 1)$, hence $x = y$.
     (b) $x$ and $y$ are both even. Then $f(x) = x + 1$ and $f(y) = y + 1$.
     Therefore, $(f(x) = f(y)) \Rightarrow x + 1 = y + 1$, so $x = y$ again.
     (c) One of the numbers is even and the other is odd. WLOG, Let $x$ be even and $y$ be odd.
     Then $f(x) = x + 1$ and $f(y) = y - 1$. Therefore, $(f(x) = f(y)) \Rightarrow (x + 1 = y - 1)$
     Observe that $x + 1$ is an odd number, while $y - 1$ is an even number; so $x + 1 = y - 1$ is a contradiction. Therefore, this case can not occur.

     Since whenever $f(x) = f(y)$ we have $x = y$, $f$ is one-one.
   - $f$ is onto:
     *Proof:*We shall show that for all $b \in \mathbb{N}$ there is $a \in \mathbb{N}$ such that $f(a) = b$. There are two cases:
     (a) $b$ is even. Then $b \in \mathbb{N} \Rightarrow b \geq 0 \Rightarrow b + 1 \in \mathbb{N}$ is odd. Therefore, $f(b+1) = (b+1) - 1 = b$, as desired.
     (b) $b$ is odd. Then $b \in \mathbb{N} \Rightarrow b \geq 1 \Rightarrow b - 1 \geq 0 \Rightarrow b - 1 \in \mathbb{N}$ is even. Therefore, $f(b-1) = (b = 1) + 1 = b$, as desired.

     Therefore, $f$ is onto, as desired.

   Since $f$ is one-one and onto, it is bijective by definition.

# 5 Problem 5

Show that the product $(a+bi)(c+di)$ of two complex numbers can be evaluated using just three real-number multiplications. You may use a few extra additions.

*Solution:*

$$\begin{aligned}(a+bi)(c+di) &= ac + (bc)i + (ad)i + (bd)i^2 \\ &= (ac - bd) + (bc + ad)i\end{aligned}$$

From the equation above, we do not need to obtain 4 distinct products $ab, bc, cd, ad$; we only need to obtain $(ac - bd)$ and $(bc + ad)$. We observe that

$$(a+b)(c+d) = ab + bc + cd + ad$$

gives us the sum of $(bc + ad)$ width $ac$ and $bd$ in one multiplication. Hence in three multiplications we compute

$$\begin{aligned}X &= (a+b)(c+d) &= ac + bc + ad + bd \\ Y &= ac & \\ Z &= bd &\end{aligned}$$

Now, without using any additional multiplications, we obtain:

$$\begin{aligned}(ac - bd) &= & &= (Y - Z) \\ (bc + ad) &= (ac + bd + bc + ad) - (ac + bd) & &= X - (Y + Z)\end{aligned}$$

as desired.

# 6 Problem 6

Given a function $f : A \to A$. An element $a \in A$ is called a *fixed point* of $f$ if $f(a) = a$. Find the set of fixed points, $S$, for each of the following functions.

1. $f\ :\ A \to A$ where $f(x) = x$

   *Solution:* The set of fixed points is $A$ by definition: $x \in A \Rightarrow f(x) = x$

2. $f\ :\ \mathbb{N} \to \mathbb{N}$ where $f(x) = x + 1$ is fixed for $x \in \emptyset$.

   *Solution:* The set of fixed points $S$ is empty: $x \in S \Rightarrow f(x) = x \Rightarrow x = x + 1$, a contradiction.

3. $f\ :\ \mathbb{N}_6 \to \mathbb{N}_6$ where $f(x) = 2x \pmod 6$ is fixed for $x \in \{0\}$.

   *Solution:* The set of fixed points is $S = \{0\}$. This can be verified by looking at the table of values the function takes throughout its domain:

| x | f(x) |
|---|---|
| 0 | 0 |
| 1 | 2 |
| 2 | 4 |
| 3 | 0 |
| 4 | 2 |
| 5 | 4 |

4. $f\ :\ \mathbb{N}_6 \to \mathbb{N}_6$ where $f(x) = 3x \pmod 6$ is fixed for $x \in \{0, 3\}$.

   *Solution:* The set of fixed points is $S = \{0, 3\}$. This can be verified by looking at the table of values the function takes throughout its domain:

| x | f(x) |
|---|---|
| 0 | 0 |
| 1 | 3 |
| 2 | 0 |
| 3 | 3 |
| 4 | 0 |
| 5 | 3 |

# 7 Problem 7

Let $f(x) = x^2$ and $g(x, y) = x + y$. Find compositions that use the functions $f$ and $g$ for each of the following expressions.

*Solution:*

1. $(x + y)^2 = (g(x, y))^2 = f(g(x, y))$
2. $x^2 + y^2 = g(x^2, y^2) = g(f(x), f(y))$
3. $(x + y + z)^2 = f(x + y + z) = f(g(x + y, z)) = f(g(g(x, y), z))$
4. $x^2 + y^2 + z^2 = f(x) + f(y) + f(z) = g(g(f(x), f(y)), f(z))$