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CSE 150 Foundations of Computer Science: Honors, Fall 2006

Assignment #2
Problems 1.1 – 1.5 due Tuesday, October 17th, 2006
Problems 2.0 – 2.4 due Tuesday, October 24th, 2006

Definition recap:

Function $f : X \rightarrow Y$ is said to be **one-to-one**, or *injective*, if for all a, b in X: $f(a) = f(b)$ if and only if $a = b$.

Function $f : X \rightarrow Y$ is said to be **onto**, or *surjective*, if for all b in Y there exists an a in X such that $f(a) = b$.

Function F is a **bijection** (or is *bijective*) if it is both one-to-one and onto (injective and surjective).

Problem 1.1
Show that $\mathbb{N} - A$ is countable, where $A = \{1, 2, 3, ..., k\}$ for some positive integer k.

Proof:

Let $X = \mathbb{N} - \{1, 2, 3..., k\}$. We show that X is countable by showing a bijection between X and N

Let $f : X \rightarrow \mathbb{N}$. Set $f(n) = n - k$. We show that f is a bijection.

First observe that f is one-to-one. This is true because if $f(a) = f(b)$ for some $a, b \in X$, then $a - k = b - k$, implying that $a = b$.

Now we show that f is onto. We need to show that for all $a \in \mathbb{N}$ there exists b such that $f(b) = a$ and b is in X. Let $b = a + k$. Then $f(b) = f(a + k) = a + k - k = a$. Since $a \in \mathbb{N}$, $a \geq 1$ and $a + k \geq k + 1$; therefore, $b = a + k$ is in X.

Thus, f is a bijection. □

Problem 1.2
Show that the even positive integers are countable.

Proof:

Let X be the set of even numbers. We show that X is countable by showing a bijection between X and N

Let $f : \mathbb{N} \Rightarrow X$. Set $f(n) = 2n$. We show that f is a bijection.

First observe that f is one-to-one. This is true because $a \neq b \Rightarrow 2a \neq 2b$ and therefore $f(a) \neq f(b)$.

Now we show that f is onto. We need to show that for all $a \in X$ there exists $b \in \mathbb{N}$ such that $f(b) = a$.
Consider $a \in X$. Since a is even, there is a natural number $b \in \mathbb{N}$ such that $a = 2b$. Then $f(b) = 2b = a$.

Thus, f is a bijection. □

Problem 1.3
Show that every subset of $\mathbb{N}$ is countable.

Proof:

First we observe that if a subset X is finite, it is countable by definition, so we assume that X is infinite. Then we show that X is countable by finding a bijection $f : \mathbb{N} \to X$.

Let $f(n)$ be defined by

$$f(n) = \min[X - \bigcup_{i=1}^{n-1} \{f(i)]$$

This function is well-defined: by our assumption X is infinite, so $[X - \bigcup_{i=1}^{n-1} f(i)]$ is always nonempty; and every nonempty subset of positive integers has a least element. We shall show that f is bijective.

Informal proof sketch: f returns the smallest elements of X one-by-one; once an element is returned, it is taken away from the set and never reconsidered. Hence f is one-one. Since we can get to any element by throwing away all the elements smaller than the given one, f is onto.

We first show that f is one-to-one (injective). Let $a \neq b$; WLOG let $a > b$ (so $1 \leq b \leq a-1$). Now

$$f(a) = min[X - \bigcup_{i=1}^{a-1} f(i)] = [X - (\bigcup_{i=1}^{b-1} f(i) \cup \bigcup_{i=b}^{a-1} f(i))]$$

Since $a > b$, we have that $\bigcup_{i=b}^{a-1} f(i)$ is nonempty and contains at least $f(b)$.

Therefore, $(\bigcup_{i=1}^{b-1} f(i) \cup \bigcup_{i=b}^{a-1} f(i))$ contains $f(b)$.

Therefore, $X - (\bigcup_{i=1}^{b-1} f(i) \cup \bigcup_{i=b}^{a-1} f(i))$ does **not** contain $f(b)$.

Since $f(a)$ is the minimum of the set that does not contain $f(b)$, it cannot be equal to $f(b)$.
Hence f is one-to-one.

Now we prove that f is onto induction on the elements of X.

Let $q \in X$. Let $P(q)$ be the predicate that there exists $n \in \mathbb{N}$ such that $f(n) = q$.

Base case: Since X is a subset of natural numbers, it has a least element x_0, and $f(1) = x_0$ by definition of f.

Inductive assumption: Assume that $P(r)$ holds for all $r \in X$ that are less then q.

Inductive step: We shall show that $P(q)$ holds. Indeed, among the elements of X that are smaller than q there should be the largest one, say p (since X is a subset of $\mathbb{N}$). Let $f(a) = p$ for some $a \in \mathbb{N}$.
We now show that $f(a+1) = q$. From definition of f:

$$f(a+1) = min[X - \bigcup_{i=1}^{a} f(i)]$$

We prove by contradiction that $f(a+1) = q$. Assume $f(a+1) \neq q$, say $f(a+1) = t, t \neq q$. Then $t < q$, otherwise q would be the smallest element in the set $X - \bigcup_{i=1}^{a} f(i)$ and would be equal to $f(a+1)$. Also $t > p$, since $t \in X - \bigcup_{i=1}^{a-1} f(i)$ as well, and the smallest element of that set is p. Hence we have $p < t < q$ and $t \in X$, which is a contradiction to our assumption that p is the largest element of X less than q.

Therefore, for all q $P(q)$ holds, and f is onto by the principle of mathematical induction.

Since f is one-to-one and onto, it is a bijection between X and $\mathbb{N}$, and $|\mathbb{N}| = |X|$ as desired.

Problem 1.4
Show that the set of all integers ($\mathbb{Z}$) is countable.

Proof:Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as follows:

$$f(n) = \begin{cases} 1-2n, & n < 0 \\ 2n, & n \geq 0 \end{cases}$$

We show that f is a bijection.

First we show that f is one-to-one. If $f(a) = f(b)$, then a and b must be either both positive or both non-negative; otherwise, $f(a)$ and $f(b)$ are of different parity. If they are both negative, then we have $1 - 2a = 1 - 2b$, and hence $a = b$. If they are both non-negative, then $2a = 2b$ and $a = b$ again. Hence f is one-to-one.

Now we show that f is onto. Let $n \in \mathbb{N}$. If n is even, then $n = 2k$ for some $k \in \mathbb{N}$, and $f(k) = 2k = n$. If n is odd, then $n = 2k + 1$ for some $k \in \mathbb{N}$, and $f(-k) = 1 - 2(-k) = 1 + 2k = n$. Therefore, f is onto.

Hence, f is a bijection between $\mathbb{N}$ and $\mathbb{Z}$, and $|\mathbb{N}| = |\mathbb{Z}|$, as desired. $\square$

Problem 1.5
Show that the set S of all *finite* subsets of a countable set X is countable.

Proof:We divide S into union of sets $C_1, C_2, ...$, where C_n is the set of all subsets of X that have cardinality n. So

$$S = \bigcup_{i=1}^{\infty} C_i$$

is a union of countable many sets. We shall show that each of C_i is countable as well, and then S is countable by Theorem 2.0.

Since X is countable, let $\{x_1, x_2, x_3, ...\}$ be an enumeration of X. Consider $f : C_i \to \mathbb{N}^i$ defined by $f(\{x_{a_1}, x_{a_2}, ..., x_{a_n}\}) = (a_1, a_2, ..., a_n)$, where $a_1 < a_2 < ... < a_n$. Now f is injective: if x and y are elements of C_i (so they are subsets of X with n elements), and $f(x) = f(y) = (a_1, a_2, ..., a_n)$, then by definition of our function $a_1, a_2, ..., a_n \in x$ and $a_1, a_2, ..., a_n \in y$. Since $a_1, ..., a_n$ are all different, and both x and y have n elements, it follows that $x = y$. S

From Theorem 2.0, $\mathbb{N}^n$ is countable for all n (by induction on n). Since f is an injective function to a countable set $\mathbb{N}^n$, by Problem 1.3, the domain is countable; so C_i is countable for all i. Since S is a union of countable many countable sets, it is countable and we are done. $\square$

Problem 2.0
Show that the following are statements are equivalent and true (Statements P and Q are *equivalent* if $P \Leftrightarrow Q$):

- $\mathbb{N} \times \mathbb{N}$ is countable
- Union of countably many countable sets is countable.
- $\mathbb{Q}$ is countable.

Proof:
First, we will show that these three statements are equivalent. First, visualize $\mathbb{N}\times \mathbb{N}$. It is a lattice of all natural numbers, which is the set of all pairs (a, b) where $a, b \in \mathbb{N}$. Now there is an injective map from $\mathbb{Q}$ onto $\mathbb{N} \times \mathbb{N}$ defined by $f(\frac{a}{b}) = (a, b)$; so countability of $\mathbb{N} \times \mathbb{N}$ implies countability of $\mathbb{Q}$.

Also we can illustrate that if $\mathbb{N}\times\mathbb{N}$ is countable, then the union of countably many countable sets, namely

$$X = \bigcup_{i=1}^{\infty} S_i, \text{ where } S_i\text{- countable}$$

is countable. We do that by providing an injective function from X to $\mathbb{N}\times\mathbb{N}$. Each element $x$in X is also an element of some set S_i, and since all S_i are countable, it is j'th element, which we can denote as S_{i_j}. If x belongs to several S_i's, we pick one with the smallest index. Now $g : X \Rightarrow \mathbb{N}\times\mathbb{N}$ defined by $g(S_{i_j}) = (i, j)$ is injective; so if $\mathbb{N}\times\mathbb{N}$ is countable, so is X.

The proofs in the other direction proceed in a similar manner.

Now we must prove that $\mathbb{N}\times\mathbb{N}$ can be counted. To count this set, consider a lattice walk starting at the origin. The walk is then defined in the following manner: $0 \to (0,0), 1 \to (1,0), 2 \to (0,1), 3 \to (0,2), 4 \to (1,1), 5 \to (2,0), 6 \to (3,0)$, and so on. This walk is a one-to-one function since the path never intersects itself. The function is also onto, since every ordered pair (x, y) is reached. This happens because the walk traverses all diagonal lines $y = c - x$, where $x, y, c \in \mathbb{N}$, and an ordered pair (a, b) lies on exactly one such line. Therefore, the walk constitutes a bijection between the naturals and

<u>Problem 2.1</u>

A submarine is moving along the integer number line at a constant speed s so that at each hour it is on an integer number. It started moving at time 0 at some position b. If t is the (whole) number of hours elapsed since the submarine started moving, then its position is given by the equation $x = st + b$, where x, s and b are integers.

You are working at Rocket Pizza delivery and you are to deliver pizza to the submarine. At each hour you can drop pizza on any number on the integer line. If the submarine is there at that time, then you have delivered the pizza and your job is done (you will be notified as soon as it happens).

The problem is that you don't know where the submarine is, you cannot see it, you don't know where it started and how fast it is moving (i.e., you don't know values of s and b - classified top secret data). The upside is that you have infinite number of pizzas.

Show that you can deliver pizza in a finite amount of time.

Solution: if you know the starting position b and the speed s, you can tell where the submarine is at any given time t by plugging in these values into the equation $x = st + b$. You don't know what s and b are, but since they are both integers, the set S of all possible pairs (s, b) is countable: indeed,

$$S = \{(s, b) | s, b \in \mathbb{Z}\} = \mathbb{Z} \times \mathbb{Z}$$

(by definition of $A \times B$), and $\mathbb{Z} \times \mathbb{Z}$ is countable from Theorem 2.0. That means that you can go through all possible pairs (s, b) so that each pair is reached in a finite time. As you go through a pair (s_t, b_t) at time t, you drop a pizza at the position $x = s_t \cdot t + b$; and when (s_t, b_t) matches the submarine starting position and speed(s, t), the pizza is delivered. Since you eventually reach the submarine's (s, t), you will successfully deliver pizza at that time.

<u>Problem 2.2</u>

You have 10000 kilograms of pickles. Pickles are 99 percent water by volume. Water comprises 100 percent of the mass of the pickle. Time goes by, and you observe that some water has evaporated. Now water comprises only 98 percent of the volume. What is the weight of the pickles now ?

Solution: Initially pickles are 99% water by volume, so the dry pickle compromises 1% of the pickle. After the water evaporates, the amount of dry pickle has not changed (only water evaporated), but now dry pickle compromises 2% of the pickle. Hence we can set up a system of equations:

$$\begin{aligned} \text{Dry Pickle} &= 1\% \text{ of original pickle amount} \\ \text{Dry Pickle} &= 2\% \text{ of new pickle amount} \end{aligned}$$

Therefore,

$$2\% \text{ of new pickle amount} \quad = \quad 1\% \text{ of original pickle amount}$$

Consequently,

$$\text{new pickle amount} \quad = \quad 50\% \text{ of original pickle amount}$$

Since we had 10000 kilograms of pickle originally, now we have only 5000 kilograms.

Some of the following problem require the use of mathematical induction. The following problem is solved for you as an example:

Problem 2.4 (a)
Show that $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

Proof: proof by induction.

Let $P(n)$ be the predicate that $2n \leq 2^n$

Base case: $2 \cdot 1 = 2^1$; hence $P(0)$ holds.

Inductive assumption: Let $P(n)$ hold for $n = k$, i.e. $2k \leq 2^k$ for some number k.

Inductive step: We show that given the inductive assumption, $P(k+1)$ holds. Observe that

$$2^{(k+1)} = 2^k \cdot 2 = 2^k + 2^k \geq 2n + 2n \geq (2n + 2) = 2(n + 1)$$

$$\Rightarrow 2(n + 1) < 2^{(n+1)}$$

(where the third step comes from inductive assumption, and the rest follows since $n \geq 1 \Rightarrow 2n \geq 2$).

Therefore, $P(n)$ holds for all n by the principle of mathematical induction, thus proving the theorem. □

Problem 2.4 (b)
Show that $1 + 3 + 5 + ... + (2n - 1) = n^2$

Proof: proof by induction.

Let $P(n)$ be the predicate that $1 + 3 + 5 + ... + (2n - 1) = n^2$

Base case: $1 = 1^2 = (2 \cdot 1 - 1)^2$; hence $P(0)$ holds.

Inductive assumption: Let $P(n)$ hold for $n = k$, i.e. $1 + 3 + 5 + ... + (2k - 1) = k^2$ for some number k.

Inductive step: We show that given the inductive assumption, $P(k+1)$ holds. Observe that

$$1 + 3 + 5 + ... + (2k - 1) + 2(k + 1) - 1 = k^2 + 2k + 1 = (k + 1)^2$$

(where the second step comes from inductive assumption, and the rest follows by simplification).

Therefore, $P(n)$ holds for all n by the principle of mathematical induction, thus proving the theorem. □

Problem 2.4 (c)
Show that $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{(n)(n+1)(2n+1)}{6}$

Proof: proof by induction.

Let $P(n)$ be the predicate that $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{(n)(n+1)(2n+1)}{6}$

Base case: $1 = \frac{1 \cdot 2 \cdot 3}{6}$; hence $P(0)$ holds.

Inductive assumption: Let $P(n)$ hold for $n = k$, i.e. $1^2 + 2^2 + 3^2 + ... + k^2 = \frac{(k)(k+1)(2k+1)}{6}$ for some number k.

Inductive step: We show that given the inductive assumption, $P(k+1)$ holds. Observe that

$$\begin{aligned} 1^2 + 2^2 + 3^2 + ... + k^2 + (k+1)^2 &= \frac{(k)(k+1)(2k+1)}{6} + (k+1)^2 \\ = \frac{(k+1)(2k^2 + k + 6k + 6)}{6} &= \frac{(k+1)(k+2)(2k+3)}{2} \\ &= \frac{(k+1)(k+2)(2(k+1)+1)}{2} \end{aligned}$$

(where the second step comes from inductive assumption, and the rest follows by simplification).

Therefore, $P(n)$ holds for all n by the principle of mathematical induction, thus proving the theorem. □