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CSE 150 Foundations of Computer Science: Honors, Fall 2006

Assignment #2
Problems 1.1 – 1.5 due Tuesday, October 17th, 2006
Problems 2.0 – 2.4 due Tuesday, October 24th, 2006

The goal of this homework is to improve your formal proof techniques and give you more insights on the issues of countability. The first exercise is solved for you as an example. Please be sure to read, understand and use the definitions below; you need to be precise when you prove any statement, and the only way to be precise is to use the definitions to show your results.

Definition recap:

Function $f : X \to Y$ is said to be **one-to-one**, or *injective*, if for all a, b in X: $f(a) = f(b)$ if and only if $a = b$.

Function $f : X \to Y$ is said to be **onto**, or *surjective*, if for all b in Y there exists an a in X such that $f(a) = b$.

Function F is a **bijection** (or is *bijective*) if it is both one-to-one and onto (injective and surjective).

Example 1.0:
Show that $\mathbb{N} - \{1\}$ is countable.

Proof:

Let $X = \mathbb{N} - \{1\}$. We show that X is countable by showing a bijection between X and N

Let $f : X \to \mathbb{N}$. Set $f(n) = n - 1$. We show that f is a bijection.

First observe that f is one-to-one. This is true because if $f(a) = f(b)$ for some $a, b \in X$, then $a - 1 = b - 1$, implying that $a = b$.

Now we show that f is onto. We need to show that for all $a \in \mathbb{N}$ there exists b such that $f(b) = a$ and b is in X. Let $b = a + 1$. Then $f(b) = f(a + 1) = a + 1 - 1 = a$. Since $a \in \mathbb{N}$, $a \geq 1$ and $a + 1 \geq 2$; therefore, $b = a + 1 =$ is in X.

Thus, f is a bijection. □

Problem 1.1
Show that $\mathbb{N} - A$ is countable, where $A = \{1, 2, 3, ..., k\}$ for some integer k.

Problem 1.2
Show that the even positive integers are countable.

Problem 1.3
Show that every subset of $\mathbb{N}$ is countable.

Hint: Use the fact that any nonempty set of positive integers has a least element. If X is a subset of $\mathbb{N}$, show that $f(n) = \min\{X - \bigcup_{i=1}^{n-1}\{f(i)\}\}$ gives the desired bijection for infinite X.

Problem 1.4
Show that the set of all integers ($\mathbb{Z}$) is countable.

Problem 1.5
Show that the set of all *finite* subsets of a countable set is countable.

Problem 2.0
Show that the following are statements are equivalent and true (Statements P and Q are *equivalent* if $P \Leftrightarrow Q$):

- $\mathbb{N} \times \mathbb{N}$ is countable
- Union of countably many countable sets is countable.
- $\mathbb{Q}$ is countable.

Hint: look at $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ (also known as *the integer lattice*). Start at the origin and try walking the grid in such a way that any point in it will be reached after a finite number of steps. One example of such walk would walking $\mathbb{Z}^2$ in a spiral starting from the origin (see Figure 1); try to come up with other kinds of walks on $\mathbb{Z}^2$ or $\mathbb{N}^2$. Then the walk will be an enumeration of the points in $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{N} \times \mathbb{N}$, since it yields a bijection $\phi(n) =$(the point you reach after n steps)

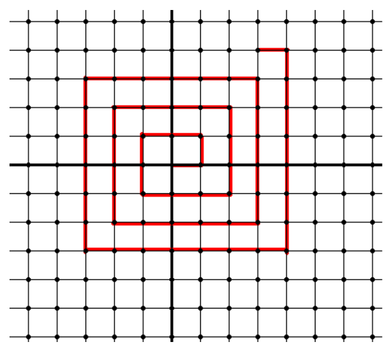

Figure 1: Lattice Walk on $\mathbb{Z} \times \mathbb{Z}$

Problem 2.1
A submarine is moving along the integer number line at a constant speed s so that at each hour it is on an integer number. It started moving at time 0 at some position b. If t is the (whole) number of hours elapsed since the submarine started moving, then its position is given by the equation $x = st + b$, where x, s and b are integers.

You are working at Rocket Pizza delivery and you are to deliver pizza to the submarine. At each hour you can drop pizza on any number on the integer line. If the submarine is there at that time, then you have delivered the pizza and your job is done (you will be notified as soon as it happens).

The problem is that you don't know where the submarine is, you cannot see it, you don't know where it started and how fast it is moving (i.e., you don't know values of s and b - classified top secret data). The upside is that you have infinite number of pizzas.

Show that you can deliver pizza in a finite amount of time.

Problem 2.2
You have 10000 kilograms of pickles. Pickles are 99 percent water by volume. Water comprises 100 percent of the mass of the pickle. Time goes by, and you observe that some water has evaporated. Now water comprises only 98 percent of the volume. What is the weight of the pickles now ?

Some of the following problem require the use of mathematical induction. The following problem is solved for you as an example:

Example 2.3
Show that $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

Proof: proof by induction.

Let $P(n)$ be the predicate that $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

Base case: $1 = \frac{1 \cdot 2}{2}$; hence $P(0)$ holds.

Inductive assumption: Let $P(n)$ hold for $n = k$, i.e. $1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}$ for some number k.

Inductive step: We show that given the inductive assumption, $P(k+1)$ holds. Observe that

$$1 + 2 + 3 + ... + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

(where the second step comes from inductive assumption, and the rest follows by simplification).

Therefore, $P(n)$ holds for all n by the principle of mathematical induction, thus proving the theorem. □

Problem 2.4
Prove the following using mathematical induction:

1. $2n \leq 2^n$
2. $1 + 3 + 5 + ... + (2n - 1) = n^2$
3. $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{(n)(n+1)(2n+1)}{6}$

The Checkerboard Problem (Bonus):
You have an $n \times n$ checkerboard with an initial set of checkers placed on it. You are allowed to add additional checkers under the following conditions: A can place checker on a square if two or more neighboring squares also have checkers on them. Neighboring cells are those above, below, to the left and to the right, as shown in Figure 2. As we showed in class, there are initial configurations of n checkers that enable the entire board to be covered. Prove that no configuration of $n - 1$ checkers can let you cover the board.

Hint: this is not induction on the number of pieces.

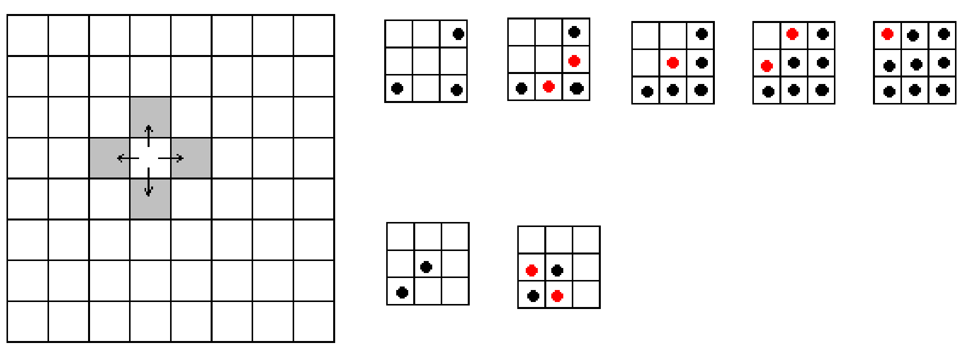

Figure 2: Neighboring cells and checkerboard evolution

Fun Bonus Problem on Function Composition

Prove that using only one operation $a * b = 1 - \frac{a}{b}$, one can get the sum, the product, the difference and the ratio of any two numbers. You may use ***only*** the two numbers a, b that are given and the operation $*$, i.e. to use a number or an operation, you have to express it first in terms of a, b and $*$.

Bonus Theorem 1.8

Show that there is no function from a set A onto $P(A)$ (so, in particular, $P(N)$ is not countable).

Hint: look at some function $\phi : A \Rightarrow P(A)$. Now for $a \in A$, $\phi(a) \subseteq A$ (why?).So for each element of A we have two cases:

1. $a \in \phi(a)$. Call such elements blue.
2. $a \notin \phi(a)$. Call such elements red.

Look at the set of all red elements. To an element of what color may this set correspond to ?