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In problems 1 - 5, find recursive relations that solve the problems. Provide a proof or an explanation for each of your answers.

Problem 1: Initially there's one pair of newborn rabbits. Rabbits become mature in one month, after which they produce a pair of newborn rabbits (male and a female) each consequent month. How many rabbits there are after n months?

Problem 2: In how many ways can you tile a $2 \times n$ checker board completely with 2×1 dominoes (so that each square is covered)?

Problem 3: In how many ways can you flip a coin n times without getting two consecutive heads?

Problem 4: An elf has a staircase of n stairs to climb. The elf likes odd numbers, so at each step he climbs either 1 or 3 stairs. In how many ways can he climb a staircase?

Problem 5: n lines are drawn on a plane such that no two of them are parallel and no three intersect at one point. In how many regions do they divide the plane into?

Hint: When you draw an additional line on the plane, how many other lines does it intersect? How many additional regions does it create?

Let $\{F_n\}$ be the *Fibonacci Sequence*. For the following problems, find formulas that evaluate the expressions stated in the problem. Prove your formulas using mathematical induction.

Problem 6: Sum of the first k even terms in $\{F_n\}$;

Problem 7: Sum of the first k odd terms in $\{F_n\}$;

Problem 8: Sum of the first k terms in $\{F_n\}$;

Problem 9: Sum of squares of the first k terms in $\{F_n\}$;

The following problems utilize the pigeonhole principle. These are the problems where you are asked to show that if there are a certain number of objects, then some number of them have some property. The steps to solve these problems are:

- ***identify the pigeons*** (the objects for which you have to show that at least a certain number of them has a certain property)
- ***identify the holes*** (a collection of classes such that all objects in the class have the desired property, and each object belongs to at least one class)

Once you have identified pigeons and holes, if there are less more pigeons than holes - you are done !

Problem 10: (Solved as an example): Three points lie on a unit segment. Show that at least two of them are within $\frac{1}{2}$ of each other.

Solution: Split the unit segment at midpoint into two segments of length $\frac{1}{2}$. Then by pigeonhole principle, one of these segments has at least two points. Since the length of the segment is $\frac{1}{2}$, the two points can be no more than $\frac{1}{2}$ apart.

Use the pigeonhole principle to prove the following statements:

Problem 11: Five points lie inside of a unit square. At least two of them are no longer than $\frac{\sqrt{2}}{2}$ apart.

Hint: In a square, the length of diagonal is the largest distance two points in a square may be apart.

Problem 12: Five points lie inside an equilateral triangle of side 1. At least two of them are less than $\frac{1}{2}$ apart.

Hint: In an equilateral triangle with side length x, the distance between any two points inside the triangle is at most x.

Problem 13: Given any 6 distinct integers from 1 to 10, their is a pair of numbers among them that has an odd sum.

Problem 14: Let n be odd and $a_1, ..., a_n$ be some permutation of $1, 2, ..., n$. Then the product $(a_1 - 1)(a_2 - 2) \dots (a_n - n)$ is even.

Problem 15: Prove that there exist two powers of 3 whose difference is divisible by 2006.

Hint: If two numbers have the same remainder when divided by n, their difference is divisible by n. How many different remainders are there?

Problem 16: Given any 7 points inside a circle of radius 1, some two of the 7 points are within 1 of each other.

Sometimes one pigeon goes to more than one hole, i.e. when you define the classes to which your object belong, one object may end up in two classes. But that is good ! That means that just need less objects for the pigeonhole principle to work.

Problem 17: (Solved as an example): Four points lie on a circle whose circumference has length 1. Show that there is an arc of length $\frac{1}{2}$ that contains at least ***three*** of them.

Solution: Place one point on a circle and draw a diameter through this point. This splits the circumference into two arcs of length $\frac{1}{2}$, *both* of which contain that point. We have three more points to add and two arcs that these points go to. By pigeonhole principle, one arc contains ***two*** of these three points, in addition to the first one - three points altogether, as desired.

Problem 18: Three points lie on a sphere. There is a hemisphere that contains all of them (hemisphere includes boundary).

Hint: You can split a sphere into two hemispheres by slicing a sphere with a plane through the center. Since the center is fixed, and a plane is defined by three points, any two points on the sphere define a plane which splits a sphere into two hemispheres. So a pair of points on a sphere split the sphere into two hemispheres.

Problem 19: Five points lie on a sphere. At least four of them lie in the same hemisphere.

Problem 20: (Induction review) Suppose that you take a piece of paper and draw n straight lines, no one exactly on top of another, that completely cross the paper. This divides the paper into polygonal regions. Prove by induction that you can color each region either red or blue so that two regions that share a boundary are always colored differently. (A single point does not count as a boundary.)

Predicate logic review

A self-proclaimed great logician has invented a new quantifier, on par with $\exists$ (there exists) and $\forall$ (for all). The new quantifier is symbolized by "$\forall^{(1)}$" and reads ***for all but one***. The proposition $\forall^{(1)}xP(x)$ is **true** iff the predicate $P(x)$ is true for all but ***exactly one*** x in the domain of the discourse. The logician has noted, "There used to be two quantifiers, but now there are three! I have extended the whole field of mathematics by 50% !"

Problem 21: Write a proposition equivalent to $\forall^{(1)}xP(x)$ using ***only*** the $\exists$ quantifier, $=$, and logical connectives $\neg$, $\vee$, and $\wedge$.

Problem 22: Write a proposition equivalent to $\forall^{(1)}xP(x)$ using ***only*** the $\forall$ quantifier, $=$, and logical connectives $\neg$, $\vee$, and $\wedge$.