

CSE 150 Foundations of Computer Science: Honors, Fall 2006

Assignment #4

Due Thursday, December 7, 2006

For of the problems below, **DO NOT** give a number as your answer. Provide an expression in terms of factorials, summations, etc., and a brief explanation for your solution.

Find the closed-form solution for the recurrences below. Initial conditions are $f_0 = 0$, $f_1 = 1$ for all problems. If you need, consult the recurrence solving guide on subsequent pages.

Problem 1:

- $f_n = 6f_{n-1} - 9f_{n-2} + 8$
- $f_n = 3f_{n-1} - 2f_{n-2} + 5$

Problem 2:

- $f_n = 5f_{n-1} - 6f_{n-2} + 5^n$
- Given a sequence defined by

$$F_n = a3^n + b5^n + c(1/2)^n$$

Find a recurrence relation that yields this sequence. Express the initial conditions in terms of a , b , and c .

Problem 3:

I have twelve books.

- In how many ways can I line them up on a single shelf?
- In how many ways can I choose seven of them and line them up on a single shelf?
- In how many ways can I choose seven to take to school?

Problem 4:

An ogre has n captives to eat, one captive per day.

1. How many ways are there to make a menu for n days ?
2. How many ways are there to pick k captives to freeze them for winter season ?
3. How many ways are there to buy n bottles of sauce for the captives, out of k kinds ?

Problem 5: Find the **recurrences** which solve part 1 and part 2 in the problem above. Don't forget initial conditions!

Problem 6:

- An ice-cream vendor sells eleven kinds of ice-cream. In how many different ways can I buy six cones, some or even all of which could be the same
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Problem 7:

- There are 33 boys, and they want to divide into three teams of eleven. In how many different ways can this be done?
- How many unique permutations exist for the letters in the 1980's band "BANANARAMA"?

Problem 8: Which number is bigger: the number of six-digit integers representable as a product of two three-digit integers, or the number of six-digit integers not representable in this form?

Problem 9: Inclusion-Exclusion

Suppose a highway survey crew noticed the following information about 500 vehicles: In 100 vehicles the driver was smoking, in 200 vehicles the driver was talking to a passenger, and in 300 vehicles the driver was tuning the radio. Further, in 50 vehicles the driver was smoking and talking, in 40 vehicles the driver was smoking and tuning the radio, and in 30 vehicles the driver was talking and tuning the radio. What can you say about the number of drivers who were smoking, talking, and tuning the radio?

Problem 10: (Extra Credit) Professors and Keys

A group of five professors are setting a mathematics competition. When they go home at night, they leave their work in a room which has a certain number of locks on the door. Each professor has keys to some, but not all of the locks. In fact, any three professors will have enough keys between them to open the door, but any two professors will not have enough. What is the smallest number of locks needed, and how many keys will each professor have? Provide a proof or a clear explanation to get credit for this problem.

A short guide to solving recurrences

Recall that in order to find a closed-form solution for a homogeneous linear recurrence

$$f_n = a_1 f_{n-1} + \dots + a_k f_{n-k}$$

(given initial conditions $f_0 \dots f_{k-1}$) you would perform the following steps:

1. Assume that there is a number r such that $f_n = r^n$ would satisfy the recurrence (ignoring initial conditions)
2. Write down the characteristic equation

$$r^k = a_1 r^{k-1} + a_2 r^{k-2} + \dots + a_{k-1} r + a_k$$

3. Solve for r (i.e., find roots of the characteristic equation)
4. If r_1, \dots, r_k are roots of the equation with multiplicity one, then

$$\begin{aligned} f_n &= r_1^n \\ f_n &= r_2^n \\ &\vdots \\ f_n &= r_k^n \end{aligned}$$

are all solutions for the recurrence. if \hat{r} is a root with multiplicity m , then

$$\begin{aligned} f_n &= \hat{r}^n \\ f_n &= n\hat{r}^n \\ f_n &= n^2\hat{r}^n \\ &\vdots \\ f_n &= n^{m-1}\hat{r}^n \end{aligned}$$

are also solutions of the recurrence

5. Observe that all linear combinations of the solutions above will constitute a solution as well
6. If there are no roots with multiplicity greater than one, all you have to do is to find constants $c_1 \dots c_k$ such

$$c_1 r_1^t + c_2 r_2^t + \dots + c_k r_k^t = f_t \tag{1}$$

for $t = 0, 1, 2, \dots, k-1$ (i.e., so that the initial conditions of the recurrence are satisfied). Then

$$f_n = c_1 r_1^n + \dots + c_k r_k^n$$

will be the desired solution. If you have roots with multiplicity greater than one, just plug their corresponding solutions into the equation (1), so that you have something of the form

$$f_n = c_1 r_1^n + \dots + c_1 \hat{r}^n + B_2 n \hat{r}^n + \dots$$

Example 1: All roots have multiplicity one

Find the closed form for the following recurrence:

$$\begin{aligned}f_n &= 4f_{n-1} - 3f_{n-2} \\f_0 &= 0 \\f_1 &= 1\end{aligned}$$

Solution: The characteristic equation for this recurrence is

$$x^2 - 4x + 3 = 0.$$

The roots of the equation are $x_1 = 1$ and $x_2 = 3$ (from quadratic formula). Hence the closed form of the recurrence will have the form

$$f_n = ax_1^n + bx_2^n = a \cdot 1^n + b \cdot 3^n.$$

All we have to do is solve for a and b so that the initial conditions are satisfied

$$\begin{aligned}f_0 &= 0 = a \cdot 1^0 + b \cdot 3^0 \\f_1 &= 1 = a \cdot 1^1 + b \cdot 3^1\end{aligned}$$

Solving for a and b , we find that $a = -1/2$ and $b = 1/2$. Hence the closed form for the recurrence is

$$F_n = -\frac{1}{2}1^n + \frac{1}{2}3^n = \frac{3^n - 1}{2},$$

as you should verify.

Example 2: Characteristic equation with roots of multiplicity greater than one
Find the closed form for the following recurrence:

$$\begin{aligned}F_n &= 4F_{n-1} - 4F_{n-2} \\f_0 &= 0 \\f_1 &= 1\end{aligned}$$

Solution: Again, the characteristic equation is

$$x^2 - 4x + 4 = 0,$$

which has a single root $x = 2$ with multiplicity 2:

$$x^2 - 4x + 4 = (x - 2)(x - 2).$$

This root yields two families of solutions: 2^n and $n2^n$. Hence the closed form for the recurrence will be

$$f_n = a2^n + bn2^n.$$

Again, all we have to do now is solve for a and b so that the solution fits the initial conditions

$$\begin{aligned}f_0 &= 0 = a \cdot 2^0 + b \cdot 0 \cdot 2^0 \\f_1 &= 1 = a \cdot 2^1 + b \cdot 1 \cdot (2^1)\end{aligned}$$

Solving for a and b , we get $a = 0$ and $b = 1/2$. So the closed form is

$$f_n = \frac{1}{2}n2^n,$$

as you should verify.

Now what if we have an inhomogeneous recurrence ?

Inhomogeneous recurrence is the one of the form

$$f_n = a_1f_{n-1} + a_2f_{n-2} + \dots + a_kf_{n-k} + \phi(n),$$

where $\phi(n)$ is some function, and $f_0 \dots f_{k-1}$ are provided.

The idea behind solving this kind of recurrence relation is that the general solution for the recurrence is, in fact, the solution to the homogeneous part plus **any** one particular solution to the whole inhomogeneous relation. This is the same idea that is used for solving differential equations.

Finding a closed-form solution for such a recurrence is usually done in three steps:

1. Ignore the $\phi(n)$ and initial conditions, and solve for the obtained homogeneous recurrence relation

$$f_n = a_1f_{n-1} + a_2f_{n-2} + \dots + a_kf_{n-k}.$$

That is, find the closed form that will look like

$$f_n = c_1r_1^n + c_2r_2^n + \dots + c_kr_k^n$$

where you know $r_1 \dots r_k$. You don't have to worry about $c_1 \dots c_k$ because they depend only on initial conditions which you are ignoring at this step.

The solution you have just obtained is called the **homogeneous solution**.

2. Now consider the original recurrence (again, ignoring the initial conditions):

$$f_n = a_1f_{n-1} + a_2f_{n-2} + \dots + a_kf_{n-k} + \phi(n)$$

Now you have to find **some** solution that satisfies this relation. This is called a **particular solution**. There are some general methods to obtain a particular solutions (using generating functions, for example), but most of the times these methods are not needed. For certain types of $\phi(n)$, intelligent guess yields a particular solution with less effort. The idea is that your solution is likely to have the same form as $\phi(n)$, but more on this later.

3. Now add the particular and homogeneous solutions together to obtain a **general solution**. Now you can solve for the constants $c_1 \dots c_k$ following the method you would use for homogeneous recurrences, i.e. solve a system of linear equations so that your closed-form solutions fits the initial conditions.

Example 3: Solving an inhomogeneous recurrence

Find the closed-form solution for the Towers of Hanoi recurrence

$$\begin{aligned} f_n &= 2f_{n-1} + 1 \\ f_0 &= 0 \end{aligned}$$

Solution: We break this problem into three steps:

1. First, we find the solution to the homogeneous part:

$$f_n = 2f_{n-1}$$

The characteristic equation for this recurrence is

$$x - 2 = 0$$

so the solution is $f_n = a2^n$, where a is some constant

2. Now we find a particular solution to the inhomogeneous recurrence. We guess that there is a solution of the form

$$f_n^* = b$$

because the inhomogeneous part has this form. If this were the case, then we have

$$\begin{aligned} b &= 2b + 1 \\ \Rightarrow b &= -1 \end{aligned}$$

So we have indeed found a solution to the recurrence: $f^*(n) = -1$

3. Now all we have to do is to add the homogeneous and the inhomogeneous solutions to obtain a general solution to the recurrence:

$$f_n = (a2^n) + (-1)$$

All we have to do now is find the right constant, a , so that our relation satisfies the initial conditions. We have:

$$f_0 = 0 = a(1) - 1$$

So $a - 1 = 0$ and $a = 1$. Therefore

$$f_n = 2^n - 1$$

is the solution to the original inhomogeneous recurrence (as you should verify).

So how do we find the particular solutions? There are several guidelines to that:

- guess that the solution to the recurrence has the same form as $\phi(n)$;
- If $\phi(n) = c$, then guess $f(n) = c$. If this doesn't work, try $f(n) = bn + c$, then $f(n) = an^2 + bn + c$, etc.
- More generally, if $\phi(n)$ is a polynomial, try a polynomial of the same degree, then a polynomial of degree one higher, then two higher, etc.
- If $\phi(n)$ is an exponential, such as 3^n , then first guess that $f(n) = c3^n$. Failing that, try $f(n) = (bn + c)3^n$, $(an^2 + bn + c)3^n$, etc.

Additional problems for practice and fun

Problem 11: Password Selection

- I have to choose a password consisting of exactly six letters (no numbers, symbols, etc). How many choices are there, if the letters can be repeated? [Hint: There are 26 letters in the alphabet!]
- Somebody hacked into my account, so I need to choose a better password. This time I will choose seven letters, but they cannot be repeated. How many choices are there? (Is this more secure?)
- Somebody hacked into my account again! This time my password will be six characters, which can be letters or numbers, and there must be at least one letter and one number. How many choices are there? (Is this more secure?)

Problem 12: Chalk Fun

- I want to bring twelve sticks of chalk to my class. There are five colors, and I want to bring at least one stick of each color. How many different combinations are there?
- I decided I'd better use white chalk too, in addition to the five colors. So I'll bring fifteen sticks altogether, at least one of each color, and at least five of which are white. How many different combinations are there?

Problem 13: Inclusion-Exclusion Find the number of all positive integers up to 1000 that are divisible neither by 2, nor by 3, nor by 5.

Problem 14: Tea fun

- Suppose that we have n teacups and m teaspoons. In how many ways one can put the teaspoons into the teacups? All teacups and teaspoons are distinguishable from each other.
- A die with n facets is rolled m times in a row. How many different outcomes are possible?
- How many maps are there from a set with m elements to a set with n elements?
- Again, n cups and m spoons, all distinguishable. How many ways are there to place spoons into cups so that each cup has **at most** one spoon ?
- How many injective maps are there from a set with m elements to a set with n elements ?
- Yet again, n cups, m spoons, all distinguishable. How many ways are there to place spoons into cups so that each cup has **at least** one spoon ? (Hint: provide an inclusion-exclusion expression)
- How many surjective maps are there from a set with m elements to a set with n elements ?
- So again, n cups and m spoons, but now the spoons are indistinguishable. How many ways are there to put these spoons into cups ?
- What if spoons are distinguishable, but the cups are not ? (Hint: the answer is a summation formula which might involve solutions from the problems above)