Logical Consequences of Formulae

- Recall: F is a logical consequence of P (i.e. $P \models F$) iff Every model of P is also a model of F.
- Since there are (in general) infinitely many possible interpretations, how can we check if F is a logical consequence of P?
- Solution: choose one "canonical" model $\Im$ such that

$$\Im \models P \text{ and } \Im \models F \Rightarrow P \models F$$

Definite Clauses

- A formula of the form $p(t_1, t_2, \ldots, t_n)$, where p/n is an n-ary predicate symbol and t_i are all terms is said to be **atomic**.
- If A is an atomic formula then
  - A is said to be a **positive literal**
  - $\neg A$ is said to be a **negative literal**
- A formula of the form $\forall(L_1 \vee L_2 \vee \cdots \vee L_n)$ where each L_i is a literal (negative or positive) is called a **clause**.
- A clause $\forall(L_1 \vee L_2 \vee \cdots \vee L_n)$ where *exactly one* literal is positive is called a **definite clause**.
  A definite clause is usually written as:
  - $\forall(A_0 \vee \neg A_1 \vee \cdots \vee \neg A_n)$
  - ... or equivalently as $A_0 \leftarrow A_1, A_2, \ldots A_n$.
- A **definite program** is a set of definite clauses.

Herbrand Universe

- Given an alphabet $\mathcal{A}$,
  the set of all *ground terms* constructed from the constant and function symbols of $\mathcal{A}$ is called the **Herbrand Universe** of $\mathcal{A}$ (denoted by $U_{\mathcal{A}}$).
- Consider the program:
  $p(zero).$
  $p(s(s(X))) \leftarrow p(X).$

  The Herbrand Universe of the program's alphabet is $\{zero, s(zero), s(s(zero)), \ldots\}$.

Herbrand Universe (contd.)

- Consider the "relations" program:

```
parent(pam, bob).        parent(bob, ann).
parent(tom, bob).        parent(bob, pat).
parent(tom, liz).        parent(pat, jim).
grandparent(X,Y) :- parent(X,Z), parent(Z,Y).
```

  The Herbrand Universe of the program's alphabet is { `pam`, `bob`, `tom`, `liz`, `ann`, `pat`, `jim` }.

Herbrand Base

- Given an alphabet $\mathcal{A}$,
  the set of all *ground atomic formulas* over $\mathcal{A}$ is called the **Herbrand Base** of $\mathcal{A}$ (denoted by $B_{\mathcal{A}}$).
- Consider the program:
  $p(zero)$.
  $p(s(s(X))) \leftarrow p(X)$.

  The Herbrand Base of the program's alphabet is
  $\{p(zero), p(s(zero)), p(s(s(zero))), \ldots\}$.

Herbrand Base (contd.)

- Consider the "relations" program:

```
parent(pam, bob).         parent(bob, ann).
parent(tom, bob).         parent(bob, pat).
parent(tom, liz).         parent(pat, jim).
grandparent(X,Y) :- parent(X,Z), parent(Z,Y).
```

  The Herbrand Base of the program's alphabet is { `parent(pam, pam)`, `parent(pam, bob)`, `parent(pam, tom)`, ..., `parent(bob, pam)`, ..., `grandparent(pam, pam)`, ..., `grandparent(bob, pam)`, ...}

Herbrand Interpretations and Models

- A *Herbrand Interprtation* of a program P is $\Im$ such that
  - $|\Im| = U_P$
  - For every constant c: $c_\Im = c$
  - For every function symbol f/n: $f_\Im(x_1, \ldots x_n) = f(x_1, \ldots, x_n)$
  - For every predicate symbol p/n: $p_\Im \subseteq (U_P)^n$
    (i.e. some subset of n-tuples of ground terms)
- A *Herbrand Model* of a program P is a Herbrand interpretation that is a model of P.

Herbrand Models

- All Herbrand interpretations of a program give the same "meaning" to the constant and function symbols.
- Different Herbrand interpretations differ only in the "meaning" they give to the *predicate* symbols.
- We often write a Herbrand model simply by listing the subset of the Herbrand base that is true in the model.
  Examples:
  - Consider our first example program.
    $\{p(zero), p(s^2(zero)), p(s^4(zero)), \ldots\}$ represents the Herbrand model that treats $p_\Im = zero, s^2(zero), s^4(zero), \ldots\}$ as the meaning of p.

Properties of Herbrand Models

1. If M is a family of Herbrand Models of a definite program P, then $\bigcap M$ is also a Herbrand Model of P.
2. For every definite program P there is a unique *least* model M_P such that
   - M_P is a Herbrand Model of P and,
   - for every Herbrand Model M, $M_P \subseteq M$.
3. For any definite program, if every Herbrand Model of P is also a Herbrand Model of F, then $P \models F$.
4. $M_P =$ the set of all ground logical consequences of P.

Sufficiency of Herbrand Models

Let P be a definite program. Then if $\Im'$ is a model of P then $\Im = \{A \in B_P \mid \Im' \models A\}$ is a Herbrand model of P.

Proof (by contradiction):

- $\Im$ is a Herbrand interpretation.
- Assume that $\Im'$ is a model but $\Im$ is not a model.
- Then there is some ground instance of a clause in P: $A_0{:-}\ A_1, \ldots, A_n$ which is not true in $\Im$
- i.e., $\Im \models A_1 \ldots \Im \models A_n$ but $\Im \not\models A_0$.
- By definition of $\Im$ then, $\Im' \models A_1 \ldots \Im' \models A_n$ but $\Im' \not\models A_0$
- Thus $\Im'$ is not a model, which contradicts our earlier assumption.

Sufficiency of Herbrand Models (contd.)

Let P be a definite program. Then if $\Im'$ is a model of P then $\Im = \{A \in B_P \mid \Im' \models A\}$ is a Herbrand model of P.

- This holds only for definite programs.
- Consider $P = \{\neg p(a), \exists X.p(X)\}$.
  - There are two Herbrand interpretations: $\Im_1 = \{p(a)\}$ and $\Im_2 = \{\}$.
  - The first is not a model of P since $\Im_1 \not\models \neg p(a)$.
  - The second is not a model of P since $\Im_2 \not\models \exists X.p(X)$
  - But there is a non-Herbrand model $\Im$:
    - $\mid \Im \mid = N$, the set of natural numbers
    - $a_\Im = 0$
    - $p_\Im =$ "is odd"

Properties of Herbrand Models

- If M_1 and M_2 are Herbrand models of P, then $M = M_1 \cap M_2$ is a model of P.
  - Assume M is not a model. Then there is some clause $A_0 :- A_1, \ldots A_n$ such that $M \models A_1 \cdots M \models A_n$ but $M \not\models A_0$.
  - Which means $A_0 \notin M_1$ or $A_0 \notin M_2$.
  - But $A_1, \ldots A_n \in M_1$ as well as M_2.
  - Hence one of M_1 or M_2 is not a model.
- There is a unique least Herbrand model.
  - Let M_1 and M_2 are two incomparable minimal Herbrand models,
  - $M = M_1 \cap M_2$ is also a Herbrand model, and
  - $M \subseteq M_1$ and $M \subseteq M_2$.
  - Thus M_1 and M_2 are not minimal.

Least Herbrand Model

The least Herbrand model M_P of a definite program P is the set of all ground logical consequences of the program.

- $M_P = \{A \in B_P \mid P \models A\}$
- First, $M_P \supseteq \{A \in B_P \mid P \models A\}$:
  - By definition of logical consequence, $P \models A$ means that A has to be in *every* model of P and hence also in the least Herbrand model.
- Second, $M_P \subseteq \{A \in B_P \mid P \models A\}$:
  - If $M_P \models A$ then A is in every Herbrand model of P.
  - But assume there is some model $\Im' \models \neg A$.
  - By sufficiency of Herbrand models, there is some Herbrand model $\Im$ such that $\Im \models \neg A$.
  - Hence A is not in some Herbrand model, and hence is not in M_P.

Finding the Least Herbrand Model

Immediate consequence operator:

- Given $I \subseteq B_P$, construct I' such that
  $$I' = \{A_0 \in B_P \mid A_0 \leftarrow A_1, \ldots A_n \text{ is a ground instance of a clause in } P \text{ and } A_1, \ldots, A_n \in I\}$$
- I' is said to be the immediate consequence of I.
- Written as $I' = T_P(I)$
  T_P is called the *immediate consequence operator*.
- Consider the sequence: $\emptyset$, $T_P(\emptyset)$, $T_P^2(\emptyset)$, …, $T_P^i(\emptyset)$, ….
- $M_P \supseteq T_P^i(\emptyset)$ for all i.
- Let $T_P \uparrow \omega = \bigcup_{i=0}^{\infty} T_P^i(\emptyset)$.
  Then $M_P \subseteq T_P \uparrow \omega$

Computing Least Herbrand Models: An Example

```
parent(pam, bob).
parent(tom, bob).
parent(tom, liz).
parent(bob, ann).
parent(bob, pat).
parent(pat, jim).

anc(X,Y) :-
   parent(X,Y).
anc(X,Y) :-
   parent(X,Z),
   anc(Z,Y).
```

M_1	$\emptyset$
$M_2 = T_P(M_1) =$	{parent(pam,bob), parent(tom,bob), parent(tom,liz), parent(bob,ann), parent(bob,pat), parent(pat,jim) }
$M_3 = T_P(M_2) =$	{anc(pam,bob), anc(tom,bob), anc(tom,liz), anc(bob,ann), anc(bob,pat), anc(pat,jim) } $\cup M_2$
$M_4 = T_P(M_3) =$	{anc(pam,ann), anc(pam,pat), anc(tom,ann), anc(tom,pat), anc(bob,jim) } $\cup M_3$
$M_5 = T_P(M_4) =$	{anc(pam,jim), {anc(tom,jim) } $\cup M_4$
$M_6 = T_P(M_5) =$	M_5

Computing M_P: Practical Considerations

- Computing the least Herbrand model, M_P, as the *least fixed point* of T_P:
  - terminates for Datalog programs (programs w/o function symbols)
  - may not terminate in general
- For programs with function symbols, computing logical consequence by first computing M_P is impractical.
- Even for Datalog programs, computing least fixed point directly using the T_P operator is wasteful (known as *Naive* evaluation).
- Note that $T_P^i(\emptyset) \subseteq T_P^{i+1}(\emptyset)$.
- We can calculate $\Delta T_P^{i+1}(\emptyset) = T_P^{i+1}(\emptyset) - T_P^i(\emptyset)$ [The difference between the sets computed in two successive iterations]
- This strategy is known as *semi-naive* evaluation.