The alphabet of predicate logic

- Variables
- Constants (identifiers, numbers etc.)
- Functors (identifiers with arity > 0; e.g. *date*/3).
- Predicate symbols (identifiers with arity ≥ 0; e.g. *append*/3).
- Connectives: $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation), $\leftrightarrow$ (logical equivalence), $\supset$ (implication).
- Quantifiers: $\forall$ (universal), $\exists$ (existential).
- Auxilliary symbols such as parantheses and comma.

Predicate Logic Formulas

Terms ($\mathcal{T}$) over an alphabet $\mathcal{A}$ is the smallest set such that:

- Every constant $c \in \mathcal{A}$ is also $\in \mathcal{T}$.
- Every variable $X \in \mathcal{A}$ is also $\in \mathcal{T}$.
- If $f/n \in \mathcal{A}$ and $t_1, t_2, \ldots t_n \in \mathcal{T}$ then $f(t_1, t_2, \ldots, t_n) \in \mathcal{T}$.

Well-formed *formulas* (*wffs*, denoted by F) over alphabet $\mathcal{A}$ is the smallest set such that:

- If p/n is a predicate symbol in $\mathcal{A}$ $t_1, t_2, \ldots t_n \in \mathcal{T}$ then $p(t_1, t_2, \ldots, t_n) \in \mathcal{F}$.
- If $F, G \in \mathcal{F}$ then so are $(\neg F)$, $(F \wedge G)$, $(F \vee G)$, $(F \supset G)$ and $(F \leftrightarrow G)$.
- If $F \in \mathcal{F}$ and X is a variable in $\mathcal{A}$ then $(\forall X\ F)$ and $(\exists X\ F) \in \mathcal{F}$.

Bound and Free Variables

- Variable X is *bound* in formula F if $(\forall X\ G)$ or $(\exists X\ G)$ is a subformula of F.
- A variable that occurs in F, but is not bound in F is said to be *free* in F.
- A formula F is *closed* if it has no free variables.
- Let $X_1, X_2, \ldots X_n$ be all the free variables in F. Then
  - $(\forall X_1\ (\ldots\ (\forall X_n\ F)\ldots))$ is the *universal closure* of F, and is denoted by $\forall F$.
  - $(\exists X_1\ (\ldots\ (\exists X_n\ F)\ldots))$ is the *existential closure* of F, and is denoted by $\exists F$.

Interpretation

An interpretation $\Im$ of an alphabet is

- a non-empty domain $\mathcal{D}$, and
- a mapping that associates:
  - each constant $c \in \mathcal{A}$ with an element $c_{\Im} \in \mathcal{D}$
  - each n-ary functor $f \in \mathcal{A}$ with an function $f_{\Im} : \mathcal{D}^n \rightarrow \mathcal{D}$
  - each n-ary predicate symbol $p \in \mathcal{A}$ with an relation $p_{\Im} \subseteq \mathcal{D}^n$

For instance, one interpretation of the symbols in our "relations" program is that `bob`, `pam` et al are people in some set, and `parent/2` is the parent-of relation, etc.
Another interpretation could be that `bob`, `pam` etc are *natural numbers*, `parent/2` is the greater-than relation, etc.

Valuation

- Given an interpretation $\Im$, the semantics of a variable-free (a.k.a. *ground*) term is clear from $\Im$ itself:

$$\Im(f(t_1, t_2, \ldots t_n)) = f_{\Im}(\Im(t_1), \Im(t_2), \ldots, \Im(t_n))$$

- But to attach a meaning to terms with variables, we must first give a meaning to its variables.
- This is done by a *valuation*: which is a mapping from variables to the domain $\mathcal{D}$ of an iterpretation.
- $\varphi = \{X_1 \mapsto d_1, X_2 \mapsto d_2, \ldots X_n \mapsto d_n\}$
- $\varphi[X \mapsto d]$ is identical to φ except that it maps X to d.

Semantics of terms

Terms are given a meaning with respect to a *valuation*

- Given an interpretation $\Im$ and valuation φ, the meaning of a term t, denoted by $\varphi_{\Im}(t)$ is defined as:
  - if t is a constant c then $\varphi_{\Im}(t) = c_{\Im}$
  - if t is a variable X then $\varphi_{\Im}(t) = \varphi X$
  - if t is a structure $f(t_1, t_2, \ldots, t_n)$ then $\varphi_{\Im}(t) = f_{\Im}(\varphi_{\Im}(t_1), \varphi_{\Im}(t_2), \ldots, \varphi_{\Im}(t_n))$

Example

Let $\mathcal{A}$ be an alphabet containing constant *zero*, a unary functor *s* and a binary functor *plus*.
$\Im$, defined as follows, is an interpretation with N (the set of natural numbers) as its domain:

- $zero_{\Im} = 0$
- $s_{\Im}(x) = 1 + x$
- $plus_{\Im}(x, y) = x + y$

Now, if $\varphi = \{X \mapsto 1\}$, then

$$\begin{aligned} \varphi_{\Im}(plus(s(zero), X)) &= \varphi_{\Im}(s(zero)) + \varphi_{\Im}(X) \\ &= (1 + \varphi_{\Im}(zero)) + \varphi(X) \\ &= (1 + 0) + 1 = 2 \end{aligned}$$

Semantics of Well-Formed Formulae

A formula's meaning is given w.r.t. an interpretation $\Im$ and valuation φ.

- $\Im \models_{\varphi} p(t_1, t_2, \ldots, t_n)$ iff $\langle \varphi_{\Im} t_1, \varphi_{\Im} t_2, \ldots, \varphi_{\Im} t_n \rangle \in p_{\Im}$
- $\Im \models_{\varphi} \neg F$ iff $\Im \not\models_{\varphi} F$
- $\Im \models_{\varphi} F \wedge G$ iff $\Im \models_{\varphi} F$ and $\Im \models_{\varphi} G$
- $\Im \models_{\varphi} F \vee G$ iff $\Im \models_{\varphi} F$ or $\Im \models_{\varphi} G$ (or both)
- $\Im \models_{\varphi} F \supset G$ iff $\Im \models_{\varphi} G$ whenever $\Im \models_{\varphi} F$
- $\Im \models_{\varphi} F \leftrightarrow G$ iff $\Im \models_{\varphi} F \supset G$ and $\Im \models_{\varphi} G \supset F$
- $\Im \models_{\varphi} \forall X\ F$ iff $\Im[X \mapsto d] \models_{\varphi} F$ for every $d \in |\Im|$
- $\Im \models_{\varphi} \exists X\ F$ iff $\Im[X \mapsto d] \models_{\varphi} F$ for some $d \in |\Im|$

Example 1.

Consider the language with *zero* as the lone constant, *s/1* as the only functor symbol, and a predicate symbol $p/1$.
Consider an interpretation $\Im$ with $|\Im| = N$, the set of natural numbers, $zero_{\Im} = 0$ and $s_{\Im}(x) = 1 + x$
Now consider the formula:

$$F_1 = p(zero) \wedge (\forall X \; p(s(s(X))) \leftrightarrow p(X))$$

Find an interpretation for $p/1$ such that $\Im \models F_1$.

Given a set of closed formulas P, an interpretation $\Im$ is said to be a *model* of P iff every formula of P is true in $\Im$

Example 2.

Recall Example 1:

$$F_1 = p(zero) \wedge (\forall X \; p(s(s(X))) \leftrightarrow p(X))$$

Consider extending the previous example with another predicate symbol $q/1$, and consider the formula:

$$F_2 = q(s(zero)) \wedge (\forall X \; q(s(s(X))) \leftrightarrow q(X))$$

Now extend the previous interpretation such that $\Im \models F_1 \wedge F_2$.

Example 3.

Recall Example 2:

$$\begin{aligned} F_1 &= p(zero) \wedge (\forall X \ p(s(s(X))) \leftrightarrow p(X)) \\ F_2 &= q(s(zero)) \wedge (\forall X \ q(s(s(X))) \leftrightarrow q(X)) \end{aligned}$$

In the previous example, consider a new formula:

$$F_3 = (\forall X \ q(s(X)) \leftrightarrow p(X))$$

Now extend the previous interpretation such that $\Im \models F_1 \wedge F_2 \wedge F_3$.

Interpretations and Consequences

- *Is there any interpretation $\Im$ such that $\Im \models F_1 \wedge F_2$, but $\Im \not\models F_3$?*

Logical Consequence

Let P and F be closed formulas.

F is a logical consequence of P (denoted by $P \models F$) iff F is true in every model of P.

Logical Consequence: An Example

1. $(\forall X\ (\forall Y\ (mother(X) \wedge child(Y, X)) \supset loves(X, Y)))$
2. $mother(mary) \wedge child(tom, mary)$

- Is $loves(mary, tom)$ a logical consequence of the above two statements?
- For (1) to be true in some interpretation $\Im$:

$$\Im \models_{\varphi} (mother(X) \wedge child(Y, X)) \supset loves(X, Y)$$

  must hold for any valuation φ.
- Specifically,

$$\Im \models_{\varphi} (mother(mary) \wedge child(tom, mary)) \supset loves(mary, tom)$$

- Hence $loves(mary, tom)$ is true in $\Im$ if (2) above is true in $\Im$.