Closure Properties

Let R and S be equivalence relations on a set A. Is the union $R \cup S$ also an equivalence relation?

Let A be the set $\{1, 2, 3\}$ and consider binary relations

$$R = \{(1,1),(1,2),(2,1),(2,2),(3,3)\}$$
and
$$S = \{(1,1),(2,2),(2,3),(3,2),(3,3)\}.$$

Both R and S are equivalence relations.

The relation $R \cup S$ contains pairs $(1, 2)$ and $(2, 3)$, but not $(1, 3)$. It is therefore not transitive and hence not an equivalence relation.

The two properties of reflexivity and symmetry are preserved under set union, but as the example shows, transitivity need not be preserved.

In such cases one often considers extensions of a given relation that satisfy certain properties. Minimal such extensions are known as "closures."

We next discuss closures of relations under reflexivity, symmetry, and transitivity.

Reflexive Closures

Let R be a binary relation on a set A.

By the *reflexive closure* of R we mean the relation

$$r(R) = R \cup E,$$

where E denotes the set $\{(x, x) : x \in A\}$.

For example, if R is the $<$ relation on the integers, then $r(<)$ is the $\leq$ relation.

Theorem.

> If R is a binary relation on A, then $r(R)$ is a reflexive binary relation with $R \subseteq r(R)$.
>
> Furthermore, whenever S is a reflexive relation on A with $R \subseteq S$, then $r(R) \subseteq S$.

The first part of the theorem follows immediately from the definition of reflexive closure.

The second part states that $r(R)$ is the smallest reflexive relation (in a set-theoretic sense) that contains R as a subset.

Corollary.

> If R is reflexive, then $r(R) = R$.

Symmetric Closures

Let R be a binary relation on a set A.

By the *symmetric closure* of R we mean the relation

$$s(R) = R \cup R^{\leftarrow},$$

where $R^{\leftarrow}$ denotes the converse of R, i.e., the set $\{(y, x) : (x, y) \in R\}$.

For example, if R is the relation $\{(1,1),(1,2),(1,3)\}$, then

> $R^{\leftarrow} = \{(1,1),(2,1),(3,1)\}$
> and $s(R) = \{(1,1),(1,2),(2,1),(1,3),(3,1)\}$.

Theorem.

> If R is a binary relation on A, then $s(R)$ is a symmetric binary relation with $R \subseteq s(R)$.
>
> Furthermore, whenever S is a symmetric relation on A with $R \subseteq S$, then $s(R) \subseteq S$.

The first part of the theorem again follows immediately from the definition of symmetric closure, and the second part states that $s(R)$ is the smallest symmetric relation that contains R as a subset.

Corollary.

> If R is symmetric, then $s(R) = R$.

Composite Relations

Let R be a relation on $A \times B$ and S a relation on $B \times C$. Then the *composite relation* $R \circ S$, or simply RS, is defined to be the set

$$\{(x, z) \in A \times C \ : \ \text{for some } y \in B,\ xRy \text{ and } ySz\}.$$

For example, if $R = \{(1, a), (2, a), (3, b), (4, c)\}$ and $S = \{(a, b), (b, a)\}$, then $RS = \{(1, b), (2, b), (3, a)\}$.

If R is a binary relation on A, i.e., a subset of $A \times A$, we define the "k-fold" composition of R by induction as follows.

$$R^k = \begin{cases} R & \text{if } k = 1 \\ R^{k-1}R & \text{if } k > 1 \end{cases}$$

Lemma.

We have $R^j \circ R^k = R^{j+k}$, for all $j, k \geq 1$.

Transitive Closures

Let R be a binary relation on a set A.

By the *transitive closure* of R we mean the relation

$$t(R) = \bigcup_{k \geq 1} R^k.$$

For example, if R is the relation $\{(1,1),(1,2),(2,3)\}$, then $t(R) = \{(1,1),(1,2),(2,3),(1,3)\}$.

Theorem.

> If R is a binary relation on A, then $t(R)$ is a transitive binary relation with $R \subseteq t(R)$.
>
> Furthermore, whenever S is a transitive relation on A with $R \subseteq S$, then $t(R) \subseteq S$.

The proof of this theorem is not as straightforward as the proofs of the corresponding theorems for reflexivity and symmetry. The second part states that $t(R)$ is the smallest transitive relation that contains R as a subset.

Corollary.

> If R is transitive, then $t(R) = R$.

Properties of Closures

Proposition

> If R is a binary relation on A, then $r(r(R)) = r(R)$, $s(s(R)) = s(R)$, and $t(t(R)) = t(R)$.

Theorem

1. If R is reflexive, then so are $s(R)$ and $t(R)$.
2. If R is symmetric, then so are $r(R)$ and $t(R)$.
3. If R is transitive, then so is $r(R)$.

This theorem implies the validity of various identities between relations. For instance, $r(s(r(R))) = s(r(R))$ or $s(r(s(R))) = s(r(R))$.

Exercise. Show that $r(s(R)) = s(r(R))$, for all binary relations R on a set A.

An important consequence of the previous results is:

Theorem

> If R is a binary relation on a set A, then $t(r(s(R)))$ is an equivalence relation on A.

Is $s(t(r(R)))$ *also an equivalence relation?*