Predicate Logic

Predicate, or first-order, logic is a formal logical systems that extends propositional logic in that it provides for additional operators, called "quantifiers, " and for variables that range over domains other than Boolean values.

For example, consider the definition of reflexivity:

> A binary relation R on a set A is called *reflexive* if xRx for all x in A.

This definition can not be represented in propositional logic, but requires predicate logic. Suitable equivalent formulations of reflexivity are

$$\forall x \; [x \in A \Rightarrow (x, x) \in R]$$

and

$$(\forall x \in A) \; (x, x) \in R.$$

The symbol $\forall$ is called a *quantifier* or more specifically a *universal quantifier*. The letter x denotes a variable, universally quantified in this example, that ranges over the elements of A.

Intuitive Semantics

The intuitive meaning of the above formulas is clear. Given a set A and a subset $R \subseteq A \times A$, either formula is true if, and only if, the set R contains all pairs (a, a), where a is an element of A.

For example, let A be the set $\{1, 2, 3\}$.

If R denotes the binary relation $\{(1, 1), (2, 2), (3, 3)\}$ then the above formulas are true. But if R denotes the set $\{(1, 2), (2, 3), (3, 1)\}$ then the formulas are false.

In other words, the truth value of the above formulas depends on how the sets A and R are interpreted. In some cases the formulas are true; in other cases, false.

Consider now a slightly more complicated formula,

$$\forall x \forall y \forall z \; [(x \in A \wedge y \in A \wedge z \in A) \Rightarrow \\ ((x, y) \in R \wedge (y, z) \in R \Rightarrow (z, x) \in R)]$$

or an equivalent, sligtly shorter formula,

$$\forall x \in A \, \forall y \in A \, \forall z \in A) \; [(x, y) \in R \wedge (y, z) \in R \Rightarrow (z, x) \in R].$$

These statements express that the relation R is *circular*. Informally, the formula is true if, for all x, y, and z in A such that xRy and yRz, one has zRx.

Taking the set $A = \{1, 2, 3\}$ and the relation

$$R = \{(1, 1), (2, 2), (3, 3)\}$$

again, we find that the formulas are true.

The formulas are also true if R denotes the set

$$\{(1,2),(2,3),(3,1)\}.$$

However, if we take the union of the two relations

$$\{(1,1),(2,2),(3,3),(1,2),(2,3),(3,1)\}$$

the formulas turn out to be false.

Informal Reasoning

It turns out that if a binary relation R on a set A is reflexive and circular, then it is an equivalence relation. That is, it is also symmetric and transitive.

In predicate logic this can be expressed by the following formula:

$$
\begin{array}{l}
(\forall x \in A)\ (x,x) \in R \wedge \\
\quad (\forall x \in A \forall y \in A \forall z \in A)[(x,y) \in R\ \wedge\ (y,z) \in R \Rightarrow (z,x) \in R] \\
\qquad \Rightarrow \\
(\forall x \in A \forall y \in A)\ [(x,y) \in R \Rightarrow (y,x) \in R] \wedge \\
\quad (\forall x \in A \forall y \in A \forall z \in A)[(x,y) \in R\ \wedge\ (y,z) \in R \Rightarrow (x,z) \in R]
\end{array}
$$

This implication happens to be true, regardless of how the sets A and R are interpreted. In other words, the formula represents a statement that is a logically valid based on its structure and the meaning of the logical operations (propositional connectives and quantifiers).

We next give an informal proof of the validity of the above formula.

An Informal, but Detailed Proof

1. An implication $\alpha \Rightarrow \beta$ can be proved by showing that β is true under the assumption that α is true.

2. Let us assume that both the reflexivity axiom,
$$(\forall x \in A)\ (x, x) \in R,$$
and the circularity axiom,
$$\forall x \in A \forall y \in A \forall z \in A)[(x, y) \in R \wedge (y, z) \in R \rightarrow (z, x) \in R],$$
are true.

3. We have to show that the symmetry axiom,
$$(\forall x \in A \forall y \in A)\ (x, y) \in R \Rightarrow (y, x) \in R,$$
and the transitivity axiom,
$$(\forall x \in A \forall y \in A \forall z \in A)[(x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R],$$
are true under these assumptions.

4. Let us first consider
$$(\forall x \in A \forall y \in A)\ [(x, y) \in R \Rightarrow (y, x) \in R].$$

5. To prove that this formula is true (under the above assumptions) we show that
$$(a, b) \in R \Rightarrow (b, a) \in R$$
is true, where a and b are arbitrary elements in A.

6. Let us assume that $(a,b) \in R$ is true.

7. We have to show that $(b,a) \in R$ is also true.

8. From reflexivity we may conclude that $(b,b) \in R$ is true.

9. Thus our assumptions imply that the conjunction

$$(a,b) \in R \wedge (b,b) \in R$$

is true.

10. Circularity implies that the implication

$$(a,b) \in R \;\wedge\; (b,b) \in R \rightarrow (b,a) \in R$$

is true.

11. By Modus Ponens we conclude that $(b,a) \in R$ is true, which completes the first part of our proof.

12. In the second part we show that

$$(\forall x \in A \forall y \in A \forall z \in A)[(x,y) \in R \wedge (y,z) \in R \Rightarrow (x,z) \in R]$$

follows from reflexivity, circularity, and symmetry (which we just proved).

13. To prove transitivity it suffices to show that

$$(a,b) \in R \;\wedge\; (b,c) \in R \Rightarrow (a,c) \in R$$

is true, where a, b, and c are arbitrary elements of A.

14. Let us assume $(a, b) \in R \ \wedge \ (b, c) \in R$ is true.

15. We have to show that then $(a, c) \in R$ is also true.

16. Circularity implies that

$$(a, b) \in R \ \wedge \ (b, c) \in R \Rightarrow (c, a) \in R$$

is true.

17. Using Modus Ponens we may conclude that $(c, a) \in R$ is true.

18. We may use symmetry to conclude that

$$(c, a) \in R \ \Rightarrow (a, c) \in R$$

is true.

19. Using Modus Ponens again we conclude that $(a, c) \in R$ is true, which completes the second part of the proof.

20. In sum, we have shown that reflexivity and circularity imply symmetry and transitivity.

A Shorter Proof

The following is a more typical proof of the above assertion. It is considerably shorter, but necessarily less formal and contains fewer details.

Let R be a reflexive and circular relation. We show that R is symmetric and transitive.

Symmetry. Suppose $(x, y) \in R$. By reflexivity we also have $(y, y) \in R$ and therefore, by circularity, $(y, x) \in R$.

Transitivity. Suppose $(x, y) \in R$ and $(y, z) \in R$. By circularity, $(z, x) \in R$ and hence, by symmetry, $(x, z) \in R$.