Predicate Versus Propositional Logic

The interaction between propositional connectives, such as implication, and quantifiers can be subtle and one has to be careful in determining the logical structure of informal statements.

For example, consider the statement that

> *there is a student in CSE-213 such that if he or she drinks, then everybody in the class drinks!*

Is this statement true?

Before we answer the question let us analyze the logical structure of the statement. That is, we will decompose the statement into simpler parts and show how these parts are combined via logical operators.

Elementary statements in predicate logic are represented via predicates.

Predicates

Propositions are sentences that can be true or false.

John takes CSE-213.
John drinks.

A *predicate* can be thought of as a "sentence with blanks," also called an *open sentence* or a *sentence schema*.

______ *takes CSE-213.*
______ *drinks.*

Note that sentence schemas do not denote statements (that can be true or false). One rather has to specify, or *substitute*, suitable values for the blanks to obtain sentences that can be true or false.

John takes CSE-213.
Mary takes CSE-213.
Mary drinks.
etc.

Predicate Formulas

Predicates represent functions that map (one or more) arguments to Boolean values.

For example, the above sentences can be represented as follows, using predicates $TakesCSE213$ and $Drinks$ (both of which take a single argument):

$$TakesCSE213(John)$$
$$TakesCSE213(Mary)$$
$$Drinks(Mary)$$

In the language of predicate logic, $John$ and $Mary$ are constants, interpreted to denote certain persons.

Variables

In addition to constants, the formal language of predicate logic also provides *variables* to denote (unspecified) elements from a specified domain.

Thus, one can also construct formulas,

$$TakesCSE213(x) \text{ or } Drinks(y),$$

which again do not denote statements (that are true or false).

We obtain statements by substituting values for variables.

For instance, if we replace the variable x by the constant $John$ or the constant $Mary$, or replace y by the constant $Mary$, then we obtain the sentences listed previously from the above formulas.

Sometimes it is necessary to consider sentence schemas with more than one blank, such as

_____ *drinks and* _____ *takes CSE-213.*

There is some ambiguity, though, as it is not clear which of the following formulas is intended:

$$Drinks(x) \wedge TakesCSE213(y)$$

or

$$Drinks(x) \wedge TakesCSE213(x)?$$

Substitution

A *substitution* is a mapping that assigns values to variables.

If one *applies* a substitution to a formula with variables, all occurrences of the same variable are simultaneously replaced by the same value. The resulting formula is also called an *instance* of the original formula.

Thus,

$$Drinks(John) \wedge TakesCSE213(Mary)$$

is an instance of

$$Drinks(x) \wedge TakesCSE213(y)$$

but not of

$$Drinks(x) \wedge TakesCSE213(x).$$

In sum, the elementary formulas in predicate logic are expressions of the form

$$P(x_1, \ldots, x_n)$$

where P is called a *predicate symbol* and $x_1, \ldots, x_n$ are called *variables*.

Formulas with variables are neither true nor false, but one can apply substitutions to obtain (possibly infinitely many) sentences (each of which is true or false).

Domains and Truth Sets

The set of values that may be substituted for a variable is called its *domain*.

Different variables may have different domains:

> x *hit more than 60 home runs in a season.*
> y *is greater than* z*.*

The domain of x is the set of all (professional) baseball players, past and present; whereas y and z denote numbers.

We are particularly interested the *truth set* of a predicate formula, that is, the set of domain elements (or tuples of domain elements, in case there is more than one variable) which yield a true statement after substitution.

For the above formulas we get:

> $\{Maris, McGwire, Sosa\}$
> $\{(m, n)\,|\,m$ is a number greater than $n\}$

Truth sets define the logical meaning of a predicate.

Relations

Formally, the meaning of predicate formulas (with one or more variables) is given by *relations*.

The meaning of a formula $P(x)$ is defined as the set

$$\{x \mid x \in D \text{ and } P(x) \text{ is true}\}$$

The meaning of a formula $Q(x, y)$ in two variables is the set of pairs

$$\{(x, y) \mid x \in D,\ y \in D',\ \text{and } Q(x, y) \text{ is true}\}$$

The meaning of a formula $R(x_1, \ldots, x_n)$ in n variables is the set of n-tuples

$$\{(x_1, \ldots, x_n) \mid x_1 \in D_1, \ldots, x_n \in D_n,\ \text{and } R(x_1, \ldots, x_n) \text{ is true}\}$$

In short, predicates are *syntactic expressions* and relations define their *semantics*.

Example

The two sentences

> *Susan Frank is teaching section 7 of CSE-213.*

and

> *Michael Steinberg is teaching section 10 of CSE-213.*

have the same logical structure,

> ______ *is teaching section* ______ *of course* ______

(but not necessarily the same truth value).

A more concise (and unambiguous) formulation is

$$Teaches(x, y, z),$$

where x denotes a person, y a section, and z a course; i.e., the three variables range over different domains.

Substitution of values for variables, as we shall see, is only one way of obtaining a (true or false) sentence from a predicate.