Quantifiers

Some student in CSE-213 drinks.
All students in CSE-213 drink.

These are propositions that can be true or false. Both statements are constructed from predicate formulas and logical operators called *quantifiers*.

In predicate logic there are two kinds of quantifiers:

- *universal quantifier* $\forall$
  (read *for all*)

- *existential quantifier* $\exists$
  (read *there exists*)

Quantifers turn a predicate into a sentence:

$$\exists x\ [TakesCSE213(x) \wedge Drinks(x)]$$

$$\forall x\ [TakesCSE213(x) \Rightarrow Drinks(x)]$$

These sentences are called *universal* and *existential statements*, respectively.

Semantics of Quantified Statements

A universal statement

$$\forall x F[x]$$

is true if, and only if, $F[d]$ is true for *all* domain elements d.

Thus, if just one person does not drink, then $\forall x\ Drinks(x)$ is false.

An existential statement

$$\exists x\ F[x]$$

is true if, and only if, $F[d]$ is true for *some* domain element d.

For instance, $\exists x\ Teaches(x, CSE213)$ is true, as there is indeed a person who teaches CSE 213.

Abbreviated Formulas

One often writes $x \in D$ to indicate the domain of a quantified variable, e.g.,

$$(\exists x \in D) Drinks(x).$$

This way of specifying the domain of a variable can be viewed as a short-hand for the formula

$$\exists x (x \in D \wedge Drinks(x)).$$

In other words, the domain of the variable x is explicitly indicated by a predicate formula, $x \in D$.

Similarly, the expression

$$(\forall x \in D) Drinks(x)$$

can be viewed as a short-hand for

$$\forall x (x \in D \Rightarrow Drinks(x)).$$

Examples

Some students do not satisfy the prerequisites for CSE-213.

$$\exists x(Student(x) \land \neg PrereqCSE213(x))$$

All students who satisfy the prerequisites for CSE-213 may take the course.

$$\forall x[Student(x) \land PrereqCSE213(x) \Rightarrow TakesCSE213(x)]$$

Some students in CSE-213 drink.

$$\exists x[TakesCSE213(x) \land Drinks(x)]$$

If some students drink then all students drink.

$$(\exists x Drinks(x)) \Rightarrow (\forall x Drinks(x))$$

$$(\exists y Drinks(y)) \Rightarrow (\forall z Drinks(z))$$

Drinking Example Continued

Recall the following statement from the previous lecture. *There is a student in CSE-213 such that if he or she drinks, then everybody in the class drinks.*

The following formalizations of this sentence have been proposed. Several proposed formalizations:

$$\exists x \, TakesCSE213(x) \Rightarrow \forall y \, Drinks(y)$$

$$\forall x[Drinks(x) \Rightarrow TakesCSE213(x) \wedge Student(x)]$$

$$\exists x[TakesCSE213(x) \wedge Drinks(x) \Rightarrow Alldrink(x)]$$

$$\exists x[TakesCSE213(x) \wedge Drinks(x) \Rightarrow \forall x \, Drinks(x)]$$

$$\begin{aligned} &\exists x[TakesCSE213(x) \wedge (Drinks(x)] \\ &\qquad \Rightarrow \forall y[TakesCSE213(y) \Rightarrow Drinks(y)] \end{aligned}$$

None of these formulas quite captures the given sentence. Here is a correct solution:

$$\begin{aligned} \exists x[\quad & TakesCSE213(x) \wedge \\ & \quad (Drinks(x) \Rightarrow \\ & \qquad \forall y(TakesCSE213(y) \Rightarrow Drinks(y)))] \end{aligned}$$

The Drinking Principle

The predicate logic formula

$$\begin{aligned}\exists x[\quad & TakesCSE213(x) \wedge \\ & \quad (Drinks(x) \Rightarrow \\ & \qquad (\forall y(TakesCSE213(y) \Rightarrow Drinks(y))))]\end{aligned}$$

is true!

To verify this we just need to find one person for which the statement within the square brackets is true. Any student who takes CSE-213 is a candidate.

There are two cases.

(1) Suppose there is a student in CSE-213 who does not drink. Let John be such a student. Then the conjunction inside the square brackets is true, because the first conjunct is $TakesCSE213(John)$ and the second conjunct is an implication with a hypothesis, $Drinks(John)$, that is false.

(2) If all the students in CSE-213 drink, then we can easily check that the conjunction within the brackets is true for any one of them, because in that case the conclusion of the conditional,

$$\forall y(TakesCSE213(y) \Rightarrow Drinks(y)),$$

is true by assumption.

In sum, the statement is true (though one student did point out that the above argument is based on the assumption that CSE 213 is popular enough to attract at least one student).

Examples of Predicate Logic Formulas

Only dogs bark.

Rephrase: *It barks only if it is a dog.*

Or equivalently: *If it barks, then it is a dog.*

$$\forall x\,[Barks(x) \Rightarrow Dog(x)]$$

Everyone has a father.

$$\forall x\,[Person(x) \Rightarrow \exists y\,(Person(y) \wedge Father(y, x))]$$

Nobody is infallible.

$$\neg\exists x\,[Person(x) \wedge Infallible(x)]$$

Negations of Quantified Statements

The negation of a universal statement is logically equivalent to an existential statement:

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$

Not all students drink.
Some student does not drink.

The negation of an existential statement is logically equivalent to a universal statement:

$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

No student failed the course.
All students passed the course.

The above equivalences indicate that only one of the two quantifiers is really needed.