Implicit Quantification

Sometimes the formal representation of a statement requires quantifiers, even though none of the telltale words "all", "some", etc. is present.

For example,

> *If a number is an integer, then it is a rational number*

looks like a conditional statement, but is more accurately formalized as a universal statement,

$$\forall x[Integer(x) \rightarrow Rational(x)]$$

Rephrasing the informal statement makes the use of a quantifier explicit.

> *Every integer is a rational number*

Existential quantification can also be implicit.

> *The number 24 can be written as a sum of two even integers.*

$$\exists m \exists n[Even(m) \wedge Even(n) \wedge 24 = m + n]$$

Domains and Predicates

There are different ways of specifying the domain of a predicate variable.

(1) Explicitly indicate the domain:

$$(\forall x \in D)\; Q(x)$$

(2) Represent the domain by a predicate:

$$\forall x[D_P(x) \rightarrow Q(x)]$$

where $D_P(x)$ is meant to be true if, and only if, x is an element of D.

A statement

$$\forall x[P(x) \rightarrow Q(x)]$$

is said to be *vacuously true* or *true by default* if $P(x)$ is false for every x.

This implies that a statement

$$(\forall x \in D)\; Q(x)$$

is true whenever the domain D is empty.

Multiple Quantifiers

Quantifiers can be nested, with alternations between universal and existential quantifiers.

> *Everybody loves somebody.*
> *Somebody loves everybody.*

These statements have similar structure, but with different order of quantifiers.

$$\forall x \exists y Loves(x, y)$$
$$\exists x \forall y Loves(x, y)$$

Such statements are often difficult to evaluate.

Are the two statements equivalent?

Does one of them imply the other?

Consider two similar statements,

$$(\forall m \in Z)(\exists n \in Z) n > m$$
$$(\exists m \in Z)(\forall n \in Z) n > m$$

where Z denotes the domain of integers.

Limits and Nested Quantifiers

Informally, a number L is the limit of a sequence

$$a_1, a_2, \ldots, a_n, \ldots$$

if the values a_n become arbitrarily close to L as n gets larger.

This concept can be formally defined in predicate logic as follows.

$$(\forall \epsilon > 0)(\exists N)(\forall n)[n > N \rightarrow L - \epsilon < a_n < L + \epsilon]$$

The logical complexity of this formula (two alternations of quantifiers) explains why most students find it hard to understand the concept of a limit.

Note that a formula

$$(\forall \epsilon > 0)F$$

is just a shorthand for

$$\forall \epsilon \, [\epsilon > 0 \rightarrow F]$$

Evaluation of Complex Formulas

The truth value of complex formulas with quantifiers may be difficult to determine.

Every even number is the sum of two primes.

$$\forall k \, [Even(k) \rightarrow (\exists m \exists n [Prime(m) \wedge Prime(n) \wedge k = m + n])]$$

$$\forall a \forall b \forall c \forall n \, [\quad (a > 0 \wedge b > 0 \wedge c > 0 \wedge n > 2) \rightarrow a^n + b^n \neq c^n]$$

The difficulty with evaluation of quantified statements, and a key difference between predicate and propositional logic, is that variables denote elements of some domain, which may be infinite.

For instance, a universal statement $\forall x \, P(x)$ is true if $P(x)$ is true for *all* possible values of x.

If the given domain is *infinite*, e.g., the set of the integers or the real numbers, there are infinitely many cases to consider!

Angels and Devils

Sometimes it's helpful to pit an *angel*, whose job it is to make a formula true, against a *devil*, who attempts to make a formula false.

The two opponents scan a given formula, the angel making a move on an existential quantifier, whereas the devil takes his turn on a universal quantifier.

Each move consists of choosing a value for the quantified variable in question, dropping the quantifier, and applying the chosen substitution to the remaining formula.

Example.

$$\forall x\ \exists y\ y > x$$

1. Faced with $\forall x\, \exists y\, y > x$, the devil chooses $x = 1{,}000$.

2. The angel gets $\exists y\, y > 1{,}000$ and chooses $y = 1{,}001$.

3. The game ends with the proposition $1{,}001 > 1{,}000$, which is true. The angel wins.

The angel wins whenever the final proposition is true. Otherwise the devil wins.

More Challenging Game

Every integer has a (integer) square root.

$$\forall x \; \exists y \; (y \times y = x)$$

1. The devil begins and chooses $x = 4$.

2. The angel gets $\exists y \, (y \times y = 4)$ and chooses $y = 2$

3. The result is a true proposition, $2 \times 2 = 4$. The angel wins.

Poor strategy on the devil's part! Another try:

1. The devil begins by choosing $x = 3$.

2. The angel gets $\exists y (y \times y = 3)$ and chooses $y = 1$.

3. The result is a false proposition, $1 \times 1 = 3$. The devil wins.

This is a winning strategy for the devil! The angel has no chance (in this case).

It's the Strategy

If there is a winning strategy for the devil, the original formula is false.

$$\forall x \; \exists y \; (y \times y = x)$$

We have seen a winning strategy for the devil: choose $x = 3$ (or $x = 5$, or $x = 19$, etc).

If there is a winning strategy for the angel, the original formula is true.

$$\forall x \; \exists y \; y > x$$

A winning strategy for the angel is to choose the number $n + 1$ for y, whenever the devil has initially chosen n for x.