Logical Equivalence and Consequence

Some relationships hold between formulas just on the basis of their syntactical form, regardless of the meaning of the predicates.

Examples are more generalized forms of the logical equivalences from propositional logic.

$$\neg\neg Drinks(x) \equiv Drinks(x)$$

$$\forall x Drinks(x) \vee \neg\forall x Drinks(x) \equiv T$$

That is, if two propositional formulas are logically equivalent, one may replace propositional formulas by arbitrary predicate logic formulas: the resulting formulas are still equivalent.

Other equivalences derive from more subtle connections between quantifiers and propositional connectives.

If F and G are sentences in predicate logic (i.e., F and G contain no free variables), then we say that F *logically implies* G (or G is a *logical consequence* of F), and write $F \vdash G$, if G is true whenever F is true.

We say that two sentences F and G are *logically equivalent*, written $F \equiv G$, if each sentence logically implies the other.

Logical Relationships in the Predicate Calculus

$$
\begin{aligned}
\forall x P(x) &\quad\vdash\quad \exists x P(x) && (1)\\
\forall x \forall y P(x,y) &\quad\equiv\quad \forall y \forall x P(x,y) && (2)\\
\exists x \exists y P(x,y) &\quad\equiv\quad \exists y \exists x P(x,y) && (3)\\
\exists x \forall y P(x,y) &\quad\vdash\quad \forall y \exists x P(x,y) && (4)\\
\neg \forall x P(x) &\quad\equiv\quad \exists x \neg P(x) && (5)\\
\neg \exists x P(x) &\quad\equiv\quad \forall x \neg P(x) && (6)\\
\exists x P(x) &\quad\equiv\quad \neg \forall x \neg P(x) && (7)\\
\forall x P(x) &\quad\equiv\quad \neg \exists x \neg P(x) && (8)\\
(\forall x P(x)) \wedge (\forall x Q(x)) &\quad\equiv\quad \forall x (P(x) \wedge Q(x)) && (9)\\
(\exists x P(x)) \vee (\exists x Q(x)) &\quad\equiv\quad \exists x (P(x) \vee Q(x)) && (10)
\end{aligned}
$$

Are the following logical equivalences valid or not?

$$
\begin{aligned}
(\forall x P(x)) \vee (\forall x Q(x)) &\quad\equiv\quad \forall x (P(x) \vee Q(x))\\
(\exists x P(x)) \wedge (\exists x Q(x)) &\quad\equiv\quad \exists x (P(x) \wedge Q(x))
\end{aligned}
$$

If they are not valid, can at least a logical consequence relation be established between the respective formulas?

A Sample Proof

We prove:

$$\exists x \forall y P(x, y) \quad \vdash \quad \forall y \exists x P(x, y)$$

We have to show that for any domain, and interpretation of predicates over that domain, in which $\exists x \forall y P(x, y)$ is true, the sentence $\forall y \exists x P(x, y)$ is also true.

Let D be a domain and suppose predicates are interpreted in such a way that $\exists x \forall y P(x, y)$ is true. By the semantics of the universal quantifier, this implies that there is an element, say d_0, in D such that $\forall y P(d_0, y)$ is true. By the semantics of the universal quantifier, this implies that $P(d_0, d')$ is true for all elements d' in D.

We have to prove that $\forall y \exists x P(x, y)$ is true, assuming the same interpretation of predicates over domain D. By the semantics of the universal quantifier, it suffices to show that $\exists x P(x, d')$ is true for all elements d' in D. This in turn can be established by showing that $\exists x P(x, d)$ is true where d is an arbitrary element in D.

We already know that $P(d_0, d')$ is true for all elements d' in D. Thus in particular $P(d_0, d)$ is true. But by the semantics of the existential quantifier this implies that $\exists x P(x, d)$ is true, which completes the proof.

Dracula

Recall that a logical argument is a sequence of statements, of which the last one is called the *conclusion*, while all the others are called *hypotheses*.

An argument is valid if the conclusion is a logical consequence of the (conjunction of all the) hypotheses.

Example.

> (1) Everyone is afraid of Dracula.
> (2) Dracula is afraid only of me.
> *Therefore* I am Dracula.

Is this argument valid?