# Sets

Sets are a basic data structure of mathematics.

Informally a set is simply a (well-defined) collection of objects.

Well-known examples of sets from mathematics include the set of integers, the set of rational numbers, the set of real numbers, etc. In computer science one often needs to deal with sets of (finite-length) strings, e.g., bitstrings, or sets of expressions that are well-formed according to certain syntax rules.

While the concept of sets is deceptively simple, all mathematical objects can, in principle at least, be described or defined in terms of sets. In that sense sets are the basic building blocks for constructing mathematical (i.e., formal, abstract) objects.

We will discuss key components of the theory of sets and later on explore various applications of sets to computing.

# Sets and Elements

Two basic concepts of set theory are the notion of a *set* and the *element relationship*.

We use the symbol $\in$ to denote the element relation and write

$$x \in A$$

to denote the proposition that $x$ *is an element of* $A$ (which may be true or false for specific choices of $x$ and $A$).

It is customary to denote sets by capital letters, such as $A$, $B$, and $C$, and elements by small letters, such as $x$, $y$, and $z$. But this can sometimes be confusing, for as we shall see sets can themselves be elements of other sets!

# Description of Sets

A *finite* set can (in principle) be described by listing its elements. For instance, we write

$$\{x_1, \ldots, x_n\}$$

to denote the set consisting of elements $x_1, \ldots, x_n$.

For example, the set

$$A = \{a, \{1\}, b, 1\}$$

has four elements and the statements

$$a \in A, \{1\} \in A, b \in A, \text{ and } 1 \in A$$

are all true.

Note that

1. sets can be elements of other sets: the set $\{1\}$ is an element of $A$; and

2. the set $\{A\}$ contains *one* element (whereas $A$ has four elements), so that $A \in \{A\}$ and $A \neq \{A\}$.

A similar notation as above is often used for *infinite sets* as well, e.g., when one denotes describes the set of natural numbers by

$$\mathrm{N} = \{0, 1, 2, 3, \ldots\}$$

or the set of odd natural numbers by

$$\{1, 3, 5, \ldots\}.$$

But the use of ellipsis imparts a certain degree of vagueness, and we will discuss a more formal way of defining the set of natural numbers in set theory.

# Equality

Two sets $A$ and $B$ are said to be *equal*, written $A = B$, if, and only if, they have the same elements.

More formally this can be expressed by:

$$(A = B) \Leftrightarrow \forall x \ (x \in A \Leftrightarrow x \in B).$$

Examples.

$$\begin{aligned} \{1, 2\} &= \{2, 1\}? \\ \{1, 2\} &= \{1, 1, 2, 2, 2\}? \\ \{1, 2, 3\} &= \{1, 1, 1, 3\}? \end{aligned}$$

Note that sets are *unordered* collections of objects, where the *multiplicities* of elements *don't matter*.

If $A$ and $B$ are finite sets containing a different number of elements, then they are obviously not equal.

# Comprehension

The description of a (finite) set via an explicit listing of its elements is a relatively crude specification formalism. One often obtains more intuitive descriptions of sets by characterizing elements via a logical property.

Let $A$ be a set and $P(x)$ be a formula in one variable. Then by

$$\{x \in A : P(x)\}$$

we denote the set that consists of all elements $x$ of $A$ for which $P(x)$ is true.

*Remark.* In formal set theory one has to introduce an axiom, called the *Principle of Comprehension*, which states that

> *for every set $A$ and formula $P$ there exists a set $B$, such that*
>
> $$\forall x \, [x \in B \Leftrightarrow (x \in A \wedge P(x))].$$

# Subsets

A set $A$ is said to be a *subset* of another set $B$, written $A \subseteq B$, if, and only if, every element of $A$ is also an element of $B$.

Examples.

$$\begin{aligned} \{1,2\} &\subseteq \{1,2,3\}? \\ \{1,1,2,2\} &\subseteq \{1,2\}? \\ \{1\} &\subseteq \{2,3,5,7\}? \end{aligned}$$

Note that $A$ is a subset of $B$ if the following formula is true:

$$\forall x \, [x \in A \Rightarrow x \in B].$$

*Lemma*. If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

*Proof*. If $A \subseteq B$ and $B \subseteq A$, then by the definition of the subset relation, every element of $A$ is an element of $B$ and every element of $B$ is an element of $A$. This means that $A$ and $B$ have the same elements, hence are equal.

∎

# Proper subsets

We say that $A$ is a *proper subset* of $B$, written $A \subset B$, if $A$ is a subset of $B$, but not equal to $A$:

$$A \subset B \Leftrightarrow (A \subseteq B \wedge A \neq B).$$

Example.

$$\{1, 2\} \subset \{1, 1, 2, 2\}?$$

Be careful about the distinction between the element relation and the subset relation.

Examples.

$$\begin{array}{rcl} 2 & \in & \{1, 2, 3\}? \\ \{2\} & \in & \{1, 2, 3\}? \\ 2 & \subseteq & \{1, 2, 3\}? \\ \{2\} & \subseteq & \{1, 2, 3\}? \\ \{2\} & \subseteq & \{\{1\}, \{2\}\}? \\ \{2\} & \in & \{\{1\}, \{2\}\}? \end{array}$$

# The Empty Set

Let $A$ be any set. How many elements are there in the set $\{x \in A : x \neq x\}$?

A set with no elements is called an *empty set*.

Theorem. If $\emptyset$ is an empty set, then $\emptyset \subseteq A$, for all sets $A$.

Proof. It is vacuously true that every element of an empty set is an element of every other set $A$. ∎

Corollary. There is at most one empty set.

Proof. Suppose $A$ and $B$ are both empty sets. By the theorem above we have $A \subseteq B$ and $B \subseteq A$, and hence $A = B$. ∎

Another postulate of formal set theory, the *Existence Axiom*, asserts that

> *there exists a set*,

which by the above considerations implies that there is an (unique) emptyset.

We use the symbol $\emptyset$, or sometimes $\{\}$, to denote the empty set.

# Examples of Sets

The (finite) set of integers between −2 and 5:

$$\{n \in \mathbf{Z} \mid -2 < n < 5\}$$

The (open) interval of real numbers between −2 and 5:

$$\{x \in \mathbf{R} \mid -2 < x < 5\}$$

The (infinite) set of even integers:

$$\{n \in \mathbf{Z} \mid \exists k\,(n = 2k)\}$$

From a general description it may not always be obvious what the elements of the set are:

$$\{(x, y, z) \in \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \mid \exists n(n > 2 \wedge x^n + y^n = z^n)\}.$$