Relations

A (binary) *relation* on sets A and B is a subset of the Cartesian product $A \times B$. In other words, the elements of a binary relation R are ordered pairs.

For example, if $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$, then $R = \{(a, 1), (b, 1), (b, 3)\}$ is a binary relation on A and B.

The sets A and B may be identical. For example, the less-than relation $<$ is a binary relation on $\mathbf{N}$ (and $\mathbf{N}$).

Every function is a binary reation in this sense. An important example of relations are (directed or undirected) graphs.

More generally, an *n-ary relation* on sets $A_1, \ldots, A_n$ is a subset of the n-fold Cartesian product $A_1 \times \cdots \times A_n$.

Reflexivity

Most of the relations we will discuss have significant "internal" structure. In particular, we will study equivalence relations and order relations.

A binary relation R on a set A is said to be *reflexive* if $(x, x) \in R$, for all $x \in A$.

For example, the less-than relation is not reflexive, but the less-than-or-equal-to relation is.

For simplicity we will often write "xRy" instead of "$(x, y) \in R$" when R is a binary relation.

A relation R on A is said to be *irrfelexive* if xRx for *no* $x \in R$.

The less-than relation is irreflexive.

Note that irreflexivity is different from non-reflexivity. Every irreflexive relation R on a non-empty set A is also non-reflexive, but a non-reflexive relation need not be irreflexive.

Transitivity and Symmetry

A binary relation R on a set A is said to be *transitive* if whenever x, y, z are elements of A with xRy and yRz, then xRz.

Both the less-than relation and the less-than-or-equal-to relation are transitive.

An example of a non-transitive relation is the parent relation. The ancestor relation, though, is transitive.

A binary relation R on A is called *symmetric* if xRy implies yRx, for all x and y in A.

The equality relation is symmetric, but the less-than relation is not.

A relation R is called *antisymmetric* if xRy and yRx imply that x and y are identical, for all x and y.

The less-than relation is antisymmetric.

A relation may be neither symmetric nor antisymmetric.

Equivalence Relations

A relation that is reflexive, transitive, and symmetric is called an *equivalence relation*.

For example, the set $\{(a,a),(b,b),(c,c)\}$ is an equivalence relation on $\{a,b,c\}$.

An equivalence relation R defines "clusters" of elements of A. More formally, we define for each element $a \in A$:

$$[a]_R = \{b \in A \ : \ aRb\}.$$

If the relation R is clear from the context, we usually write $[a]$ instead of $[a]_R$.

Recall that a *partition* of a set A is a subset Π of $\mathcal{P}(A)$ such that (i) $\emptyset$ is not an element of Π and (ii) each element of A is in one, and only one, set in Π.

For example, $\{\{a\},\{b\},\{c\}\}$ and $\{\{a,c\},\{b\}$ are partitions of $\{a,b,c\}$, but $\{a\},\{b\},\{b,c\}\}$ is not.

Partitions and Equivalence Relations

Theorem.

> If R is an equivalence relation on a non-empty set A, then the equivalence classes of R constitute a partition of A.

Proof. Given in class.

Conversely, if Π is a partition of A, then

$R = \{(a, b)$: a and b are elements of the same set in $\Pi\}$

is an equivalence relation.

In other words, there is a one-to-one correspondence between equivalence relations on A and partitions on A.

For example, let $\sim$ be the relation on **N** defined by:

$$m \sim n \Leftrightarrow m + n \text{ is even.}$$

Then $\sim$ is an equivalence relation that partitions the set of natural numbers into two subsets—the sets of even and odd natural numbers, respectively.