Partial Orders

A binary relation on a set A is called a *partial order* if it is reflexive, transitive, and anti-symmetric.

Examples of partial orders are the less-than-or-equal-to relation (on the integers), the divisibility relation (on the integers), and the subset relation (on a powerset).

The symbol $\preceq$ is often used to denote partial orders.

If R is a partial order on A, one also speaks of a *partially ordered set* (A, R).

Note that a set, say the integers, can be partially ordered in different ways, e.g., by the less-than-or-equal-to relation or the divisibility relation.

Exercise.

> Is the empty set $\emptyset$ a partial order on a non-empty set A?
>
> Is the universal set $A \times A$ a partial order on A?

Quasi-Orders

The less-than relation on the integers and the proper subset relation are not partial orders, but are so-called "quasi-orders."

An irreflexive and transitive relation is called a *quasi-order*.

There is a natural correspondence between the two kinds of orders in that for every partial order one can define a corresponding quasi-order (by removing the "equality part" of the relation) and vice versa.

Lemma.

> If $\preceq$ is a partial order on a set A, then the relation $\prec$, defined by:
>
> $$x \prec y \text{ if and only if } x \preceq y \text{ and } x \neq y,$$
>
> is a quasi-order.
>
> If $\prec$ is a quasi-order on a set A, then the relation $\preceq$, defined by:
>
> $$x \preceq y \text{ if and only if } x \prec y \text{ or } x = y,$$
>
> is a partial order.

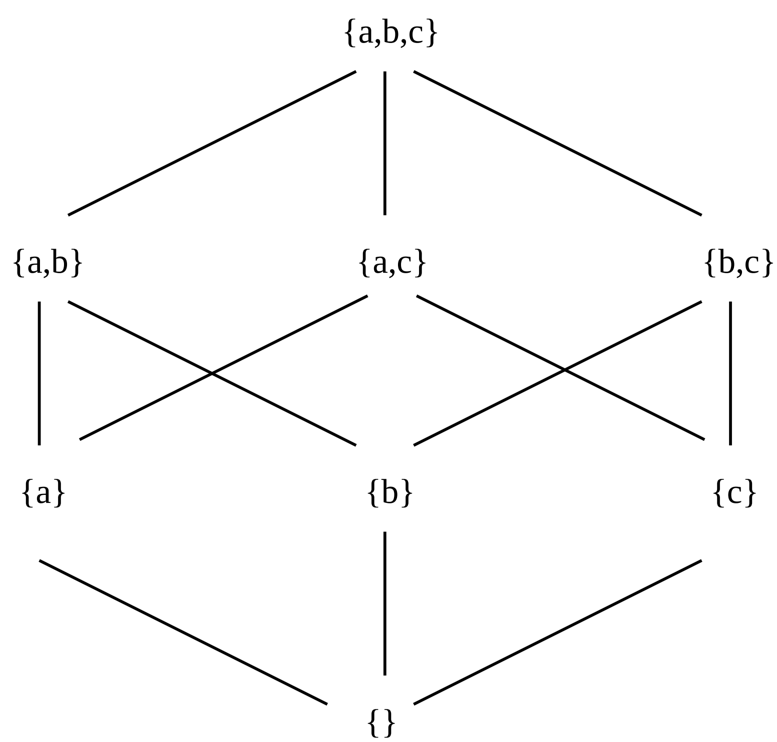
Hasse Diagrams

Partial orders, especially on finite sets, can often be conveniently represented by graphs called "Hasse diagrams."

If $\preceq$ is a partial order on A, we say that an element $y \in A$ *covers* an element $x \in A$ if, and only if $x \prec y$ and there is no element $z \in A$, such that $x \prec z$ and $z \prec y$. (Here $\prec$ denotes the quasi-order corresponding to $\preceq$.)

By the *Hasse diagram* of a partially ordered set $(A, \preceq)$ we mean the directed graph that has set of nodes A and contains as edges all pairs (x, y), such that x covers y.

Hasse diagrams are usually drawn with their edges directed downwards (and with arrow heads left off).



Hasse Diagrams (cont.)

It can be proved (see the textbook, p. 550) that every finite partially ordered set can be represented by a Hasse diagram.

Infinite partially ordered sets may or may not be representable by Hasse diagrams.

For example the set of integers with the usual $\leq$ relation can be represented by a (infinite) Hasse diagram.

But "dense" orders such as the set of rational numbers with the $\leq$ relation have no Hasse diagram, as no rational number covers any other rational number in the sense defined above. (Density refers to the property of an order that for all elements x and y with $x \prec y$, there exists an element z with $x \prec z \prec y$.)

Minimal and Maximal Elements

Let $(A, \preceq)$ be a partially ordered set.

An element $x \in A$ is said to be *maximal* (with respect to $\preceq$) if there is no $y \in A$ such that $x \prec y$; and *minimal* (with respect to $\preceq$) if there is no $y \in A$ such that $y \prec x$.

For example, the set $\{a, b, c\}$ is a maximal element with respect to the subset relation on $\mathcal{P}(\{a, b, c\})$, whereas the empty set is a minimal element.

The natural number 0 is a minimal element in the set of natural numbers with the $\leq$ relation, but there is no maximal element with respect to this partial order.

Maximal and minimal elements correspond to nodes at the top and bottom, respectively, in a Hasse diagram.

Thus, every finite partially ordered set has minimal and maximal elements. But infinite sets, such as the set of integers with the $\leq$ relation, may have neither minimal nor maximal elements.

Minima and Maxima

Let again $(A, \preceq)$ be a partially ordered set.

We say that x is a *maximum*, or *largest element*, of A if $y \leq x$, for all $y \in A$.

Similarly, x is called a *minimum*, or *smallest element*, of A if $x \leq y$, for all $y \in A$.

The empty set is a minimum, and the set A a maximum, on the partially ordered set $(\mathcal{P}(A), \subseteq)$.

By definition, a minimum has to be a minimal element, and a maximum a maximal element. But a minimal element need not be a minimum, nor a maximal element a maximum.

For example, consider the divisibility relation on the finite set $\{1, 2, 3, 4, 5, 6\}$. There is a minimum, the number 1, but no maximum. (The corresponding Hasse diagram has three top elements, 4, 5, and 6, which are mutually incomparable, $4 \not| 5$, $5 \not| 6$ and $6 \not| 4$.)

Lower and Upper Bounds

Let $(A, \preceq)$ be a partially ordered set and S be a subset of A.

If $x \preceq y$, for all $y \in S$, then x is called a *lower bound of* S; and a *greatest lower bound* if $w \preceq x$ for every other lower bound w of S.

If $y \preceq x$, for all $y \in S$, then x is called an *upper bound of* S; and a *least upper bound* if $x \preceq w$ for every other upper bound w of S.

For example, take the partially ordered set $(\mathbf{P}, |)$, where $|$ denotes the divisibility relation. Then the least upper bound of a two-element set $\{m, n\}$ is simply the least common multiple of the two integers m and n, whereas the greatest lower bound of $\{m, n\}$ is the greatest common divisor of m and n.

A *lattice* is a partially ordered set in which each subset $\{x, y\}$ has a least upper bound and a greatest lower bound.

Linear Orders

A partial order $\preceq$ is said to be *linear*, or *total*, if for all elements x and y in A, either $x \preceq y$ or $y \preceq x$ (or both).

The partial order $\leq$ on the integers is linear, but the subset relation on a $\mathcal{P}(A)$ is not linear for most sets A.

> *For which sets A is the subset relation on $\mathcal{P}(A)$ linear?*

Linear orders are also called *chains*.

A chain $(A, \preceq)$ is said to be well-ordered if every non-empty subset of A has a smallest element with respect to $\preceq$.

For example, the set of natural numbers is well-ordered under the $\leq$ relation. This property is also known as the *well-ordering principle* (of the natural numbers).

The set of integers is not well-ordered under the same relation, though.