Product Orders

There are various ways in which partial orders can be extended to product sets (i.e., sets of tuples).

Let $(A, \preceq_1)$ and $(B, \preceq_2)$ be partially ordered sets. The corresponding *product order* is defined as follows:

$$(x, y) \preceq (x', y') \text{ if and only if } x \preceq_1 x' \text{ and } y \preceq_2 y'.$$

For example, take $A = B = \mathrm{N}$ with the usual leq order. Then $(5, 5) \leq (7, 5)$, since $5 \leq 7$ and $5 \leq 5$. But $(5, 2) \not\leq (100, 1)$, because $2 \not\leq 1$.

Theorem.

> The product order of two partially ordered sets is a partial order.

Proof left as exercise.

Product orders can also be defined on n-fold cross products $A_1 \times A_2 \times \cdots \times A_n$:

$$(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n) \text{ iff } x_i \preceq_i y_i \text{ for } i = 1, \ldots, n$$

where $(A_i, \preceq_i)$ is a partially ordered set, for $i = 1, \ldots, n$.

Filing Orders

Note that a product order need not be a chain, even when its component orders are chains. There is a different way of combining orders so that the combination of chains will always result in a chain.

Let $(A_1, \preceq_1), \ldots, (A_n, \preceq_n)$ be partially ordered sets. We define a *filing relation* $\prec$ on the n-fold cross product $A_1 \times A_2 \times \cdots \times A_n$ by:

$$(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n) \text{ iff}$$
$$x_1 = y_1, \ldots, x_{k-1} = y_{k-1} \text{ and } x_k \prec_k y_k$$
$$\text{for some } k \text{ with } 1 \leq k \leq n$$

For example, we have $(5,5) \prec (5,7)$ and also $(5,5) \not\prec (4,100)$. Do we have $(4,100) \prec (5,5)$?

Another example of this relation is an arrangement of personal files by last name (first) and first name (last).

Theorem.

1. Each filing relation is a quasi-order.

2. If the component orders are chains, then the filing order is also a chain.

If the n partially ordered sets $(A_i, \preceq_i)$ are all the same, say $(A, \preceq)$, the corresponding filing order is denoted by the symbol $\prec^n$.

Orders on Formal Languages

We next consider orders defined on formal languages Σ^*.

First note, again, that

$$\Sigma^* = \bigcup_{k \in \mathbf{N}} \Sigma^k.$$

We can define orders on each set Σ^k via the filing order. More specifically, let $\preceq$ be a partial order on Σ. We define an order on Σ^k by:

$$a_1 \cdots a_k \preceq_k b_1 \cdots b_k \text{ iff } (a_1, \ldots, a_k) \preceq^k (b_1, \ldots, b_k)$$

For example, if $\Sigma = \{a, b\}$ and $a \prec b$, then $aa \prec ab \prec ba \prec bb$.

The two orders $\preceq_k$ and $\preceq^k$ are evidently closely related and we will use the same symbol, $\preceq^k$, to denote both of them.

The *standard order* on Σ^* is defined by:

$$v \preceq^* w \text{ if either } |v| < |w| \text{ or else } |v| = |w| \text{ and } v \preceq^{|v|} w$$

For the example above we have,

$$\lambda \prec^* a \prec^* b \prec^* aa \prec^* \cdots bb \prec^* aaa \prec^* \cdots$$

Lexicographic Orders

The *lexicographic relation* on Σ^* is defined by:

$a_1 \cdots a_m \prec_L b_1 \cdots b_n$ iff
either $a_1 \cdots a_k \prec^k b_1 \cdots b_k$
or else $k = m < n$ and $a_1 \cdots a_k = b_1 \cdots b_k$

where k is the minimum of m and n.

The arrangement of words in a dictionary is one example of a lexicographic order.

Theorem.

Each lexicographic relation is a quasi-order.

Taking again $\Sigma = \{a, b\}$ with $a \prec b$, we have

$$\lambda \prec_L a \prec_L aa \prec_L aaa \prec_L \cdots$$

Theorem.

If $(\Sigma, \preceq)$ is a chain, then $(\Sigma^*, \preceq^*)$ is a well-ordered chain and $(\Sigma^*, \preceq_L)$ is a chain.

The chain $(\Sigma^*, \preceq_L)$ is not well-ordered, though. For instance, there is an infinite "decreasing" chain,

$$b \succ_L ab \succ_L aab \succ_L aaab \succ_L \cdots$$

where $\succ_L$ denotes the inverse relation to $\prec_L$.