Powersets

There are various operations that allow one to construct new sets from given ones.

If A is a set, we denote by $\mathcal{P}(A)$ the set whose elements are the subsets of A.

Example. If A is the set $\{1, 2, 3\}$, then

$$\begin{aligned}\mathcal{P}(A) \quad = \quad & \{\emptyset, \\ & \{1\}, \{2\}, \{3\}, \\ & \{1, 2\}, \{1, 3\}, \{2, 3\}, \\ & \{1, 2, 3\} \\ & \}\end{aligned}$$

Do we have $1 \in \mathcal{P}(A)$, or $2 \in \mathcal{P}(A)$, or $3 \in \mathcal{P}(A)$?

No, because $1 \neq \{1\}$, etc.

In formal set theory, the existence of these sets requires another axiom, the *Powerset Axiom*:

For every set A there exists a set B, such that

$$\forall x \, (x \in B \Leftrightarrow x \subseteq A).$$

The Size of Powersets

If $A = \emptyset$, then

$$\mathcal{P}(A) = \{\emptyset\} \neq \emptyset.$$

Observation.

> $\mathcal{P}(A) \neq \emptyset$, for all sets A.

If $A = \{x\}$, then $\mathcal{P}(A) = \{\emptyset, A\}$.

If $A = \{x, y\}$, then $\mathcal{P}(A) = \{\emptyset, \{x\}, \{y\}, A\}$.

If A has n elements, how many elements are there in its powerset?

Lemma.

> If A is a set with n elements, then $\mathcal{P}(A)$ has 2^n elements.

Proof. By mathematical induction on the number of elements in A.

Further Set Operations

Other operations for constructing sets include

- *set union*
- *set intersection*
- *relative complementation* (or *set difference*)
- *complementation*

They are defined as follows.

Let A and B be subsets of some set S. We define:

$$\begin{aligned} A \cup B &= \{x \in S \,|\, x \in A \vee x \in B\} \\ A \cap B &= \{x \in S \,|\, x \in A \wedge x \in B\} \\ B - A &= \{x \in S \,|\, x \in B \wedge x \notin A\} \\ A^c &= \{x \in S \,|\, x \notin A\} \end{aligned}$$

For example, let

S be the set of real numbers,
A the set $\{x \in \mathbf{R} \,|\, -1 < x \leq 0\}$,
B the set $\{x \in \mathbf{R} \,|\, 0 \leq x < 1\}$.

What are $A \cup B$, $A \cap B$, $B - A$, and A^c?

Note that set difference can be defined as follows:

$$A - B = A \cap B^c.$$

Properties of Set Operations

Theorem.

1. $A \cap B \subseteq A$ and $A \cap B \subseteq B$
2. $A \subseteq A \cup B$ and $B \subseteq A \cup B$
3. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof (of first property).

> Let A and B be arbitrary sets. We prove that $A \cap B$ is a subset of A. By the definition of the subset relation, it suffices to show that every element of $A \cap B$ is an element of A. Let x be an arbitrary element of $A \cap B$. By the definition of intersection, we have $x \in A$ and $x \in B$. Thus x is an element of A. ∎

Set Identities

Review the following identities between sets and observe their similarity to equivalences in propositional logic.

1. Set union and intersection are commutative.

2. Set union and intersection are associative.

3. Distributivity: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

4. Double complement: $(A^c)^c = A$.

5. Idempotency: $A \cap A = A \cup A = A$.

6. De Morgan's Laws:

   $$(A \cup B)^c = A^c \cap B^c$$

   and

   $$(A \cap B)^c = A^c \cup B^c.$$

7. Absorption: $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$.

Distributivity

Theorem. For all sets A, B, and C,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Proof. Let A, B, and C be arbitrary sets. We show that the two sets $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ have the same elements:

$$\begin{aligned}
&x \in A \cap (B \cup C) \\
&\quad \text{iff} \quad x \in A \text{ and } x \in B \cup C \\
&\quad \text{iff} \quad x \in A \text{ and } (x \in B \text{ or } x \in C) \\
&\quad \text{iff} \quad (x \in A \text{ and } x \in B) \\
&\qquad\qquad\qquad \text{or } (x \in A \text{ and } x \in C) \\
&\quad \text{iff} \quad x \in A \cap B \text{ or } x \in A \cap C \\
&\quad \text{iff} \quad x \in (A \cap B) \cup (A \cap C)
\end{aligned}$$

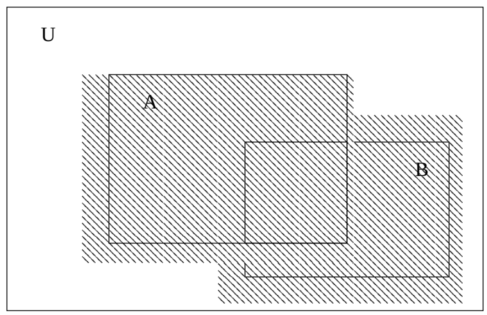
∎

Note the close connection between the “algebra of sets” and the “algebra of propositions” (Boolean algebra).

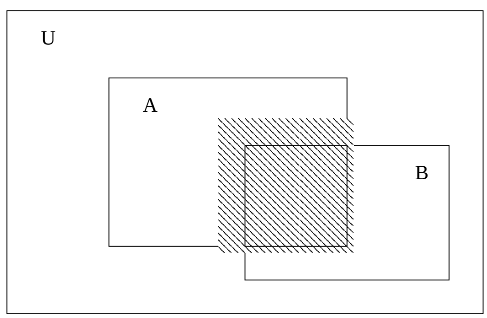
Venn Diagrams

Sets can often be conveniently represented by *Venn diagrams*.

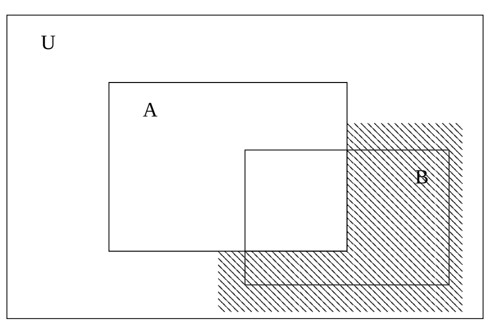
The union $A \cup B$ of A and B is represented by:



The intersection $A \cap B$ is represented by:



The set difference $B - A$ is represented by:



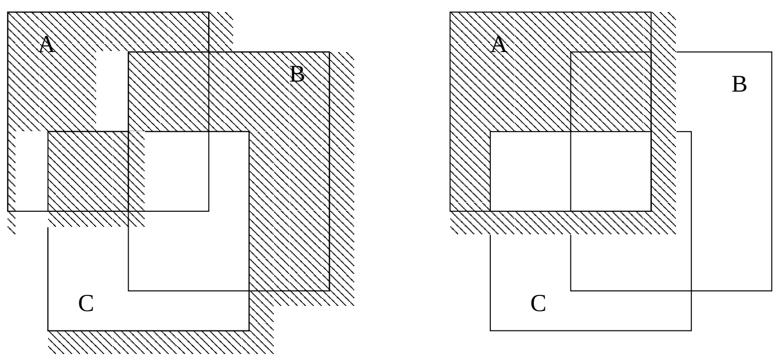
Counterexamples for Set Identities

Claim. For all sets A, B, and C,

$$(A - B) \cup (B - C) = A - C.$$

Is this claim true?

Consider the two Venn Diagrams:



The diagram on the left represents $(A - B) \cup (B - C)$, the one on the right, $A - C$.

The difference in the diagrams suggests a counterexample to the claim.

Take $A = \{x, y\}$, $B = \{y, z\}$, and $C = \{x, w\}$. Then $(A - B) \cup (B - C) = \{x, y, z\}$, whereas $A - C = \{y\}$.