Encoding the Natural Numbers in Set Theory

If A is a set, then the set $A \cup \{A\}$ is called the *successor set* of A. Sometimes the successor set of A is denoted by A'.

The natural numbers can be encoded via successor sets:

$$\begin{array}{rcccccl} 0 & = & & & \emptyset & & \\ 1 & = & 0' & = & \{\emptyset\} & = & \{0\} \\ 2 & = & 1' & = & \{\emptyset, \{\emptyset\}\} & = & \{0,1\} \\ 3 & = & 2' & = & \ldots & = & \{0,1,2\} \\ \vdots & & & & & & \\ n+1 & = & n' & = & \ldots & = & \{0,1,\ldots,n\} \end{array}$$

In other words, we can view each natural number as an abbreviation for a certain set!

One of the postulates of formal set theory asserts that there exists a set that contains the empty set and also contains the successor of each of its elements. That is, the existence of the set of natural numbers is assumed as an axiom.

Ordered Pairs and Tuples

Sets are *unordered* collections of elements.

Pairs, or more generally *tuples*, are *ordered* collections of elements.

Examples.

$$\begin{aligned} (1,2) &\neq (2,1) \\ \{1,2,3\} &= \{1,3,2\} \\ (1,2,3) &\neq (1,3,2) \\ \{1,2\} &= \{1,2,2\} \\ (1,2) &\neq (1,2,2) \end{aligned}$$

Surprisingly, (ordered) pairs can be defined in terms of (unordered) sets.

In set theory, an ordered pair (x, y) is taken as an abbreviation for the set $\{\{x\}, \{x, y\}\}$.

With this definition, do we indeed have

$$(x, y) = \{\{x\}, \{x, y\}\} \neq \{\{y\}, \{y, x\}\} = (y, x)?$$

What if $x = y$?

Tuples can be thought of as "nested" pairs. For example, we may regard $(1, 2, 3, 4, 5)$ as an abbreviation for $(1, (2, (3, (4, 5))))$ or $((((1, 2), 3), 4), 5)$.

Tuples of different length are never the same.

Number Sets

Common sets of numbers, such as the integers or the rational numbers, can be defined in terms of the natural numbers.

For instance, *integers* can be formally defined as pairs (σ, n) of a sign σ and a natural number n. There are two signs, usually written as $+$ and $-$ (and formally represented by two different sets, say $\emptyset$ and $\{\emptyset\}$).

These pairs are usually written as $+n$ (or simply n) and $-n$. There is only one 0, that is, $+0$ and -0 are considered equal.

The set of all integers is denoted by **Z**.

The set of *rational numbers* can be defined by

$$\mathbf{Q} = \{(m, n) : m \in \mathbf{Z}, n \in \mathbf{Z}, \text{ and } m \neq 0\}.$$

Rational numbers are usually written as $\frac{m}{n}$ or m/n.

Integers can be identified with rational numbers of the form $\frac{k}{1}$.

Cartesian Products

Pairs and tuples provide us with a way of constructing new sets from given ones. This will be useful when we define “functions” and “relations.”

If A and B are sets, then by $A \times B$ (read “A cross B”), we denote the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

More formally,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

The set $A \times B$ is also called the *Cartesian* (or *cross*) *product* of A and B.

For example, if $A = \{1, 2\}$ and $B = \{4, 5\}$, then

$$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5)\}.$$

Note that A and B may be the same set.

For instance, if $A = \{1, 3\}$, then

$$A \times A = \{(1, 1), (1, 3), (3, 1), (3, 3)\}.$$

If A contains m elements and B contains n elements, how many elements are there in $A \times B$?

Properties of Cartesian Products

Lemma.

> If A is a set of m elements and B a set of n elements, then $A \times B$ contains $m * n$ elements.

If $A = B$, then $A \times B = B \times A = A \times A$.

But if $A \neq B$, then $A \times B \neq B \times A$.

For example, let A be the set $\{1\}$ and B the set $\{2\}$. Then $A \times B = \{(1,2)\}$ and $B \times A = \{(2,1)\}$.

Also note that

$$A \times \emptyset = \emptyset \times A = \emptyset.$$

Lemma. For all sets A, B, and C we have

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

Proof. We need to show that $A \times (B \cup C)$ and $(A \times B) \cup (A \times C)$ have the same elements.

$$\begin{array}{ll} (x,y) \in A \times (B \cup C) & \\ \quad \text{iff} & x \in A \land y \in B \cup C \\ \quad \text{iff} & x \in A \land (y \in B \lor y \in C) \\ \quad \text{iff} & (x \in A \land y \in B) \lor (x \in A \land y \in C) \\ \quad \text{iff} & (x,y) \in A \times B \lor (x,y) \in A \times C \\ \quad \text{iff} & (x,y) \in (A \times B) \cup (A \times C) \end{array}$$

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Disjoint Sets

Two sets A and B are said to be *disjoint* if they have no elements in common, i.e., $A \cap B = \emptyset$.

Examples.

Is $\{\emptyset, \{\emptyset\}\} \cap \{\emptyset\} = \emptyset$?
No, $\{\emptyset, \{\emptyset\}\} \cap \{\emptyset\} = \{\emptyset\}$.

Is $\{\emptyset, \{\emptyset\}\} \cap \emptyset = \emptyset$?
Yes, the intersection $A \cap \emptyset$ of any set with the empty set is the empty set.

A *partition* of a set A is a collection of pairwise disjoint sets $A_1, \ldots, A_n$, such that

$$A = A_1 \cup A_2 \cup \cdots \cup A_n.$$

For example, at the end of the semester I will partition the class into subsets with grades of A, $A-$, etc. It will be a partition, since each student gets one, and only one, grade.

Partitions are closely related to equivalence relations, which we will discuss later in the semester.

Russell's Paradox

The barber of a small town agreed, for a handsome fee, to shave all the (male) inhabitants of the town who did not shave themselves, and never shave any inhabitant who did shave himself. The fee was to be paid at the end of each year.

But when the barber tried to collect the fee at the end of the first year the mayor refused to make any payment, pointing out that the barber had shaved himself and therefore violated the rule of never shaving any inhabitant who did shave himself.

Therefore the next year the barber did not shave himself. But at year-end the mayor turned him down again, pointing out that ...

... this time he had failed to shave someone who did not shave himself.

This paradox illuminates one of the pitfalls one has to avoid when setting up a formal theory of sets. For instance, allowing set operations that are too general may result in inconsistencies or contradictions in the theory.

A Set Paradox

Consider the set of all sets that are not elements of themselves:

$$S = \{A \mid A \notin A\}.$$

Is S an element of itself?

We have

$$S \in S \text{ if and only if } S \notin S,$$

which is a contradiction!

But note that the above definition of S is not covered by the Comprehension Principle. By this principle we can only define, for *some given set U*, the set

$$S = \{A \in U \mid A \notin A\}.$$

Now, if $S \in S$, then by the (new) definition of S, we get $S \notin S$, which would of course be a contradiction.

Therefore we may conclude that $S \notin S$, in which case we may *also infer* $S \notin U$. We obtain no contradiction, though.

In short, contradictions are avoided by the additional condition $A \in U$ required by comprehension.