Sequences

A *sequence* is a function the domain of which is either the set of all natural numbers (an *infinite* sequence), or some initial segment thereof (a *finite* sequence), where by an initial segment we mean any set of the form $\{0, 1, \ldots, n\}$.

For example, the infinite sequence

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$$

can be thought of as a description of a function f that maps each natural number n to a rational number, $f(n) = (-1)^n/(n+1)$.

The finite sequence

$$2, 3, 5, 7, 11, 13$$

is formally a function $f : \{0, 1, 2, 3, 4, 5\} \to \mathrm{N}$ such that

$$\begin{aligned} f(0) &= 2 \\ f(1) &= 3 \\ f(2) &= 5 \\ f(3) &= 7 \\ f(4) &= 11 \\ f(5) &= 13 \end{aligned}$$

Notations for Sequences

There is no standard notation for sequences. In the case of finite sequences one may simply list the elements in order, usually enclosed within parentheses

$$(s_1, s_2, \ldots, s_n)$$

or square brackets

$$[s_1, s_2, \ldots, s_n].$$

Curly brackets are used to denote sets, never sequences.

Similar notations are also used for infinite sequences.

For example, we write

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

to denote the *Fibonacci sequence*.

In a previous lecture we have seen a formal definition of this function by an (recursive) ML program. The n-th term in the above sequence is simply the value obtained by applying this function to argument n.

The n-th term of a sequence s is usually denoted by s_n or $s(n)$ or sometimes $s[n]$.

Note that sequences are similar to tuples, but different in terms of their set-theoretic definition.

Sequences of Sets

The domain of a sequence must be the set of natural numbers or a subset thereof. But the codomain can be any set; for instance, a set of sets.

For example, we may define a function D on the natural numbers by

$$D(n) = \{m \in \mathbf{Z} \,:\, m \text{ is a multiple of } n\}.$$

What is the codomain of this function?

The powerset of **Z***.*

There are generalized union and intersection operations that are often applied to sequences with such codomains.

Generalized Union and Intersection

If s is a sequence of sets we define two operations as follows:

$$\bigcup_k s = \{x \ : \ x \in s_k \text{ for some } k \in \mathbf{N}\}$$

and

$$\bigcap_k s = \{x \ : \ x \in s_k \text{ for all } k \in \mathbf{N}\}.$$

For example, if D is the above sequence then $\bigcup_k D = \mathbf{Z}$ and $\bigcap_k D = \{0\}$.

More generally, if A is a set, we define the *union over* A by

$$\bigcup A = \{x \ : \ x \in a \text{ for some } a \in A\}$$

and the *intersection over* A by

$$\bigcap A = \{x \ : \ x \in a \text{ for all } a \in A\}.$$

Let A be any set.

What set is $\bigcup \mathcal{P}(A)$?

The set A*.*

What set is $\bigcap \mathcal{P}(A)$?

The empty set.

Data Types based on Sequences

Arrays, lists, and strings are some examples of data types that are based on the concept of a (finite) sequence in that the values of these types represent *ordered* collections of objects.

These types are characterized by the operations that can be used to create and manipulate values of the type.

For instance, ML provides a simple notation for lists (of elements of the same type).

For example, the lists of the first three prime numbers (in increasing order) is represented by

```
- [2,3,5];
val it = [2,3,5] :  int list
```

The empty list is denoted by `[ ]`.

Polynomials

Polynomials are expressions of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0, \ldots, a_n$ are numbers (called the *coefficients*) in a specified domain and x is a variable. More specifically, one also speaks of a polynomial in x.

For example, $P_1 = 5x^2 + 3x + 2$ and $Q_1 = x^3 + 5x + 2$ are polynomials in x (with integer coefficients).

The coefficient domain is usually the set of integers or the set of real numbers, but can also be some other set on which "addition" and "multiplication" operations are defined.

A polynomial expression defines a unary function in that it can be evaluated for each specific value of x.

Thus, we have $P_1(0) = Q_1(0) = 2$, $P_1(1) = 10$, $Q_1(1) = 7$, etc.

A polynomial, and hence also its associated function, is completely determined by the coefficients (and their domain). Polynomials are therefore represented as sequences

$$(a_0, a_1, \ldots, a_n)$$

(e.g., of integers or reals). The *degree* of a polynomial is the largest index j for which $a_j \neq 0$.

For instance, P_1 is a polynomial of degree 2 represented by the sequence $(2, 3, 5)$ and Q_1 a polynomial of degree 3 represented by $(2, 5, 0, 1)$.

Polynomial Addition

The basic arithmetical operations can be extended from coefficients to polynomials as follows.

Let $P = \sum_{i=0}^{m} a_i x^i$ and $Q = \sum_{i=0}^{n} b_i x^i$ be polynomials. (Sums also provide a convenient notation for polynomials.)

The addition of P and Q is a polynomial $R = P + Q$ defined by

$$R = \sum_{i=0}^{k} (a_i + b_i) x^i.$$

where $k = max(m, n)$ and a_i is taken to be zero whenever $i > m$ and b_j is taken to be zero whenever $j > m$.

For example,

$$P_1 + Q_1 = x^3 + 5x^2 + 8x + 4.$$

Polynomial Multiplication

One can also multiply a polynomial by a number (called a scalar in this context):

$$aP = \sum_{i=0}^{m} (a * a_i) x^i.$$

Thus $3P_1$ denotes the polynomial $15x^2 + 9x + 6$.

The definition of multiplication between polynomials is slightly more complicated:

$$P \times Q = \sum_{i=0}^{m*n} c_i x^i,$$

where

$$c_i = \sum_{j=0}^{i} a_j * b_{i-j}$$

and again a_j is taken to be zero whenever $j > m$ and b_k is taken to be zero whenever $k > n$.

For example,

$$\begin{aligned} P \times Q &= 5x^5 + 3x^4 + 27x^3 \\ &\qquad + 25x^2 + 16x + 4 \end{aligned}$$