# CSE 213 - Fall 2000
## Solutions to Recommended Exercise Set 4

1. Give examples of strings in, and not in, the following sets, where $\Sigma$ denotes the alphabet $\{a, b\}$ (and $w^R$ denotes the reverse of $w$).

   (a) $L_1 = \{w \ : \ \text{for some } u \text{ in } \Sigma\Sigma,\ w = u(u^R)u\}$

   $$aaaaaa \in \Sigma,\ abbaab \in \Sigma,\ a \notin \Sigma,\ bbbbbbb \notin \Sigma$$

   (b) $L_2 = \{w \ : \ ww = www\}$

   $$e \in \Sigma,\ a \notin \Sigma,\ bb \notin \Sigma$$

   where $e$ denotes the empty string. Note that $L_2 = \{e\}$.

   (c) $L_3 = \{w \ : \ \text{for some } u,\ v \text{ in } \Sigma^*,\ uvw = wvu\}$
   We have $L_3 = \Sigma^*$ as $uvw = wvu$ whenever $u = v = e$.

   (d) $L_4 = \{w \ : \ \text{for some u in } \Sigma^*,\ uu = www\}$

   $$e \in \Sigma,\ aaaaaa \in \Sigma,\ a \notin \Sigma,\ ab \notin \Sigma$$

2. Prove the following:

   (a) For any alphabet $\Sigma$ and language $L$ over $\Sigma$, $(L^*)^* = L^*$.

   *Proof.* First note that for any language $L$ we have $L \subseteq L^*$. (This can easily be proved by using the definition of $L^*$.) Therefore we have $L^* \subseteq (L^*)^*$.
   We also need to prove $(L^*)^* \subseteq L^*$. Let $w$ be a string in $(L^*)^*$. By the definition of the Kleene star operation, there exist strings $w_1, \ldots, w_k$ in $L^*$ such that $w = w_1 \cdots w_k$. Since each string $w_i$ is in $L^*$, it can be written as $w_i = w_i^1 \cdots w_i^{i_k}$, for some strings $w_i^1, \ldots, w_i^{i_k}$ in $L$. But this implies that $w$ can be written as the concatenation,

   $$w_1^1 \cdots w_1^{i_1} w_2^1 \cdots w_2^{k_2} \cdots w_n^1 \cdots w_n^{n_k},$$

   of strings in $L$ and hence is an element of $L^*$.

   (b) For any language $L$, $L\emptyset = \emptyset L = \emptyset$.

   *Proof.* Suppose $L\emptyset$ is not empty, but contains an element, say $w$. Then $w = uv$, for some string $u \in L$ and some string $v \in \emptyset$. This contradicts the fact that the empty set contains no elements.
   The same argument applies to $\emptyset L$.

3. What language is represented by the regular expression $(((a^*a)b) \bigcup b)$?

   The set of strings that have a $b$ at the end, preceded by zero or more $a$'s.

4. Simplify the following regular expressions:

   (a) $((a^*b^*)^*(b^*a^*)^*)^*$

   The expression $(a \bigcup b)^*$ describes the same language, namely $\Sigma^*$.

   (b) $(a \bigcup b)^*a(a \bigcup b)^*$

   The expressions $b^*a(a \bigcup b)^*$ and $(a \bigcup b)^*ab^*$ describe the same language.

5. Describe the following sets by regular expressions:

   (a) All strings over $a, b$ with no more than three $a$'s.

   One possible expression is $b^*(e \bigcup a)b^*(e \bigcup a)b^*(e \bigcup a)b^*$.

   (b) All strings over $a, b$ with exactly one occurrence of the substring $aaa$.

   One possible expression is $(b \bigcup ab \bigcup aab)^*aaa(b \bigcup ba \bigcup baa)^*$. The intuition is that the (single) occurrence of $aaa$ can be (i) preceded by any string in which each occurrence of $a$ or $aa$ is followed by a $b$ and (ii) followed by any string in which each occurrence of $a$ or $aa$ is preceded by a $b$.

6. Determine whether the following statements are true:

   (a) The string $baa$ is in $L(a^*b^*a^*b^*)$.

   Take $u = e$, $v = b$, and $w = aa$. Then $u \in L(a^*)$, $u \in L(b^*)$, $v \in L(b^*)$, $w \in L(a^*)$, and hence $uvwu \in L(a^*b^*a^*b^*)$. Since $uvwu = baa$, the statement is true.

   (b) The intersection of $L(a^*b^*)$ and $L(b^*c^*)$ is empty.

   This statement is false, as $L(a^*b^*) \cap L(b^*c^*) = L(b^*)$. For instance, $bb \in L(a^*b^*)$ and $bb \in L(b^*c^*)$.