FALL'07 CSE 213: HW4

1) R_1 is not a partial ordered relation, since it is not antisymmetric. Having (1,3) and (3,1) in the relation; does not imply that 1=3 because 1≠3.
   $R_2$is a partial order because it is antisymmetric and transitive.

2) In order for a relation to be partial order, it has to be antisymmetric and transitive. Then, for uRv where length(u)<= length(v), we have to check the antisymmetric and transitive properties.
   R is not antisymmetric and we prove this by a counter example:
   Let have two strings s,t; s=a and t=b. Then, sRt and tRs are in the relation since length(s)<=length(t) and length(t)<=length(s) since both strings are of length 1. However this does no imply that s=t (according to the definition of antisymmetric relation). Then we conclude that R is not anti-symmetric, hence, not a partial order.

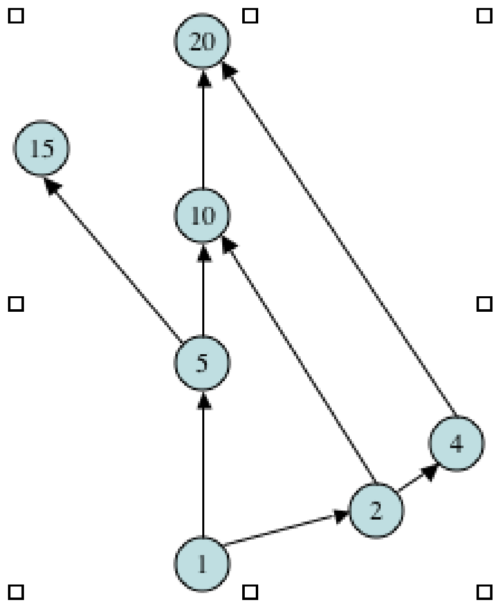
3) Relations on Z (+ -)
   a) For all m, n in Z, mR_1n iff every prime factor of m is a prime factor of n.
      R_1 is not a partial order because it is not antisymmetric
      Proving by counter example:
      mR_1n where every prime factor of m=2, (1,2) is a prime factor of n=4
      m=2 with prime factors 1 and 2
      n=4 with prime factors 1,2
      Similarly, **nR_1m** is in the relation, because every prime factor of 4 is a prime factor of 2. Howerver, n≠m because 4≠2.

   b) For all m, n in Z, mR_2n iff m+n is even.
      R_2 is not partial order because it is not antisymmetric.
      Proving by counter example:
      $3R_21$ is in the relation because 3+1 is even; also $1R_23$ is in the relation because 3+1=4 is even. However, 3≠1.

4) Divides relation on: A={1,2,4,8,…,2^n }
   In order to prove that the divides relation on A is a total order, we have to prove that A is a patial order and that every pair of elements in the divides relation, a and b are comparable; that is to say that either aRb or bRa is an element of the relation.
   Proving partial order:
   Transitivity: If (a,b) is an element of the divides relation then we know that **a** divides **b**; similarly if (b,c) is and element of the divides relation, then we know that **b** divides **c**. Transitivity holds because for elements (a,b), (b,c) ; (a,c) will be also part of the relation and the divides property will still hold.
   Antisymmetry: If (a,b) is a member of the divides relation, then (b,a) would be a member only if a=b.
   Show that aRb or bRa (a and b are comparable for all a and b):
   Let a and b, be particular but arbitrarily chosen elements of A. By definition of A, there are nonnegative integer r and s such that $a=2^r$ and $b=2^s$ (Since r, s are the exponents, nonnegative integers). Now either r < s or s < r;
   If r<s, then $b = 2^s= 2^r.2^{s-r} = a . 2^{s-r}$ where s-r >= 0. It follows, by definition of divisibility, that a divides b.

By a similar argument, if s < r, then b divides a. Hence aRb or bRa , where R is the divides relationship.

5) Hasse diagram:



Greatest: None
Least: 1 (it is minimal and 1 is less than equal all the elements in A)
Maximal: 20 and 15
Minimal: 1 (this element that has not predecessors)
Chains of length 3: 1-2-4-20 ; 1-5-10-20;
Least upper bound: 60 (last common multiple of 20 and 15)
Greatest Lower bound: 1

6) All partial order relations on A = { a, b, c}, where a is maximal
R_1= {(a,a),(b,b),(c,c)}
R_2={(a,a),(b,b),(c,c),(b,a)}
R_3={(a,a),(b,b),(c,c),(c,a)}
R_4={(a,a),(b,b),(c,c),(b,a), (c,a)}
R_5={(a,a),(b,b),(c,c),(c,b), (c,a)}
R_6={(a,a),(b,b),(c,c),(b,c), (b,a)}
R_7={(a,a),(b,b),(c,c),(c,b), (b,a),(c,a)}
R_8={(a,a),(b,b),(c,c),(b,c), (b,a),(c,a)}
R_9={(a,a),(b,b),(c,c),(b,c)}
R_{10}={(a,a),(b,b),(c,c),(c,b)}

7) Lexicographic order
a) $<_L$ if is a partial order then it has to be transitive and antisymmetric.
Recall the definition of lexicographic relation $<_L$ on Σ^*(see class notes)

$x<_L y$ is in the relation if x is a proper prefix of y or x=up,y=ur have a longest common prefix u and m is a predecessor of n.
Given $y<_L z$ also in the relation, we know that either y is a proper prefix of z or y=ur, z=us where p is a predecessor of r.
Then, $x<_L z$ will be in the relation because if x is a proper prefix of y, then x is a proper prefix of z. In addition, if x had a common long prefix with y and y had a common substring with z, then x and z have a common prefix and p is a predecessor of s. This proves transitivity.

$x<_L y$ is antisymmetric.
By contradiction, $x<_L y$ is not antisymmetric then $y<_L x$ is a member and x≠y.
However, this will never be the case because $y<_L x$ will never be in the set since x has to be a proper substring of y and y's length will have to be at least one more than x's length. Therefore we know that $y<_L x$ is not a member;

b) Prove sketch
Prove partial order for ∑* based on ∑ (Basically, for the two cases of $x<_L y$; x will be prefix of y as in $<_L$ define for ∑; and if x=wm, y=wv then w is the longest prefix in common and m is a predecessor of v.
Show that for this relation either $x<_L y$ or $y<_L x$ (i.e. they are comparable) for the cases that x is a proper prefix of y and when x and y have a long prefix in common.

8) Lexicographic order: aaa, aaab, aab, ab, abb, abba, abbba,ba,bba,bbb
Standard order: aa, ba, aaa, aab, abb, bba, bbb, aaab, abba, abbba

9) Let f(x,y) be the Ackerman function defined as:
f(x,y) = if x = 0 then y+1
else if y=0 then f(x-1,1)
else f(x-1,f(x,y-1))
Probing by well-founded induciton
Let (x,y) be an element og the well founded set NxN.
Now, lets prove that f(x,y) is defined for all (x,y) ∈ NxN. We do it in two steps:

a) Base Case:
prove that f(x,y) is true for all minimal elements (x,y) ∈ NxN
the minimal element is (0,0)
The Ackerman function is defined for (0,0) returning 0+1=**1** for the condition x=0
Therefore, it holds for the base case,

b) Let us chose an arbitrary element, (m,n) of NxN and assume that the Ackerman function is given for all predecessor of (m.n) :
f(m-1,n) ------(1)
f(m,n-1) -------(2)
f(m-1,n-1)------(3)

Then we prove that it works for f(m,n)
f(m,n)=

cases:

if m = 0 then n+1 ; since we chose a value for m,n in NxN, we know that that n is define, therefore n+1 is also defined.

else if n=0 the f(m-1,1) ; based on our assumption (3), we know that f(m-1,n) is defined for any arbitrary value of n; in this case n=1

else f(m-1,f(m,n-1)) ; based on our assumption (2), we know that f(m,n-1) will be defined and return a natural value s; then, f(m—1,s) is also define as in (1) for any n=s.

10) To prove that the relation < on NxN is well founded, we need to prove that there are not infinite descending chains or sequences.

For (a,b) < (c,d) if an only if max {a,b} < max{c,d}

By contradiction: (a,b)<(c,d) is not well-founded; then it must have a infinite descending chain. Since (a,b) < (c,d) is defined in terms of max{a,b} < max {c,b} then, it should be the case that max{a,b} < max {c,b} has an infinite descending chain also. However, since the relation is applied to NxN, we can have for instance a chain max{0,0)}< max{0,1}<max(1,2)< …. ; but there is not predecessor of (0,0). Therefore max{a,b} < max{c,d} is well founded. Since we obtained the latter based on our assumption, we conclude by contradiction the (a,b) < (c,d) does not have an infinite descending chain and therefore is well-founded.

11)

a) Let S_n denote the sum 1.3 + 2.4+3.5+...+n(n+2) . To prove tha S_n= n(n+1)(2n+7)/6

Base case: n=1, S_1 = 1.3 = 3

(1)(2)(2+7)/6 = 3

Induction Step: Assume that $S_n$is true for all n >= 1 .

Writing the sum for S_{n+1} we have:

S_{n+1} = 1.3+2.4+3.5+...+n(n+2)+(n+1)(n+3)

S_{n+1} = S_n + (n+1)(n+3).

Replacing S_n

S_{n+1}= n(n+1)(2n+7)/6 + (n+1)(n+3).

S_{n+1}= (n+1)($2n^2$ + 7n + 6n + 18]/6

S_{n+1}= (n+1)[$2n^2$ +13n +18]/6

S_{n+1}= (n+1)(n+2)[2(n+1) + 7]/6

Which is the same formula as the inductive step S_{n+1}, proving this way S_n

b) **Base case**: n=5

n^2=25 ; 2^n = 32; since 25 < 32 then base case holds

Induction Step: Assume $n^2 < 2^n$ for n > 4

Then for n+1

$(n+1)^2 < 2^{(n+1)}$

Consider $(n+1)^2 = n^2 + 2n + 1$

Using our assumption, $(n+1)^2 < 2^n + 2n + 1 < 2^n + 2^n < 2^{n+1}$. We can write this because for $n > 4$, we have $2n+1 < 2^n$. Therefore we have proved the claim that $n^2 < 2^n$ for $n > 4$.

c) Let $S_n = 1+2+4+8+\ldots+2^{n-1}$. To prove that $S_n=2^n-1$
Base Case: n=1. We just have one term in the sum and that is 1. From the formula we get 2^1-1, which is 1. Therefore, base case holds.
Induction Step: Assume that $S_n = 2^n-1$. Now, $S_{n-1}=1+2+4+8+\ldots+2^{n-1}+2^n$. Using S_n, we have $S_{n+1}=S_n+2^n=2^n-1+2^n=2^{n+1}-1$, thus we prove that $S_n=2^n-1$