Specification of Languages by Rules

We have specified the set S of balanced strings of parentheses via a definition by recursion. An alternative definition can be formulated via so-called *rules*, as follows:

$$\begin{array}{lcl} B & \to & \lambda \\ B & \to & (BB) \\ B & \to & (B) \end{array}$$

Formally, a rule is a pair of strings over an alphabet $V \cup \Sigma$.

In this example, $V = \{B\}$ and $\Sigma = \{(,)\}$.

Only elements of V may occur on the left-hand side of a rule. They are also called *nonterminals*. One of the nonterminals, in the example B, is distinguished as the *start symbol*.

The rules may be used to *generate* strings over Σ by beginning with the start symbol and then repeatedly replacing nonterminals by corresponding right-hand sides until a string in Σ^* is obtained.

For example, from B we may obtain (BB), then $((B)B)$ and $((\lambda)B)$ and finally $((\lambda)\lambda)$, which is equal to $(())$.

Context-Free Grammars

Let Σ be an alphabet.

A *(context-free) grammar* G for Σ, with start symbol S, is a finite subset of $V \times (V \cup \Sigma)^*$, where V is a set disjoint from Σ and $S \in V$.

The elements of G are called *rules* and are written as $A \to_G u$ or $A \to u$. Elements of V are called *nonterminals*; elements of Σ, *terminals*.

The set of rules in the above example is a context-free grammar. Other examples of such grammars are G_1, consisting of rules

$$\begin{array}{lcl} S & \to & \lambda \\ S & \to & SS \\ S & \to & (S) \end{array}$$

and G_2, consisting of two rules

$$\begin{array}{lcl} S & \to & \lambda \\ S & \to & (S)S \end{array}$$

In both cases, S is the start symbol (and the only nonterminal).

Languages and Replacement

We say that a string v can be obtained from u by *replacement* with G, and write $u \Rightarrow_G v$ or $u \Rightarrow v$, if there exist strings x and y in $(V \cup \Sigma)^*$ and a rule $A \to w$ in G such that $u = xAy$ and $v = xwy$.

For example, we have

$$((S)S) \Rightarrow_{G_1} ((\lambda)S)$$

and also

$$(()S) \Rightarrow_{G_1} (()\lambda).$$

A sequence of replacement steps

$$u_0 \Rightarrow_G u_1 \Rightarrow_G \cdots \Rightarrow_G u_n$$

is called a *derivation* in G of u_n from u_0.

For example,

$$S \Rightarrow SS \Rightarrow S(S) \Rightarrow S((S)) \Rightarrow S(()) \Rightarrow (S)(()) \Rightarrow ()(())$$

and

$$S \Rightarrow SS \Rightarrow (S)S \Rightarrow ()S \Rightarrow ()(S) \Rightarrow ()((S)) \Rightarrow ()(())$$

are both derivations in G_1.

We also write $u \Rightarrow_G^* v$ if v can be derived from u in this way (by zero or more replacement steps).

Thus,

$$S \Rightarrow_{G_1}^* ()(()).$$

Grammars as Language Generators

The *language* $L(G)$ *generated by* G is defined to be the set

$$\{w \in \Sigma^* : S \Rightarrow_G^* w\},$$

where S is the start symbol of G.

In other words, the language generated by a grammar is the set of all strings of terminals that can be derived from the start symbol.

For instance, the language generated by G_1 is the set S_1 and the language generated by G_2 is the set S_2. Consequently, $L(G_1) = L(G_2)$.

We will sketch a proof that $L(G_2) = S_2$ below, but first give some examples of grammars for specific languages.

Examples

Let Σ be the alphabet $\{a,b\}$. Give grammars for the following languages.

1. $L = \Sigma^*$
2. $L = \emptyset$
3. The set of all strings in Σ^* of even length.
4. $L = \{a^n b^n : n \in \mathbf{N}\}$
5. The set of all palindromes in Σ^*.
6. The set of all strings in Σ^* with an equal number of a's and b's.
7. The set of all decimal strings that represent numbers divisible by three. (In this case $\Sigma = \{0, 1, \ldots, 9\}$.)
8. The set of all strings in Σ^* with an even number of a's.

Example

Give a grammar for the set of all decimal strings that represent numbers divisible by three.

The grammar below is based on the following *observation*:

> An integer is divisible by three if the sum of its digits is divisible by three.

Let Σ be the set of digits $\{0, 1, \ldots, 9\}$ and G be the grammar with start symbol S_0 and all rules,

$$S_i \to d$$

where $d \in \Sigma$, $i \in \{0, 1, 2\}$, and d mod $3 = i$, as well as all rules

$$S_i \to dS_j$$

where $d \in \Sigma$, $i \in \{0, 1, \}$, $j \in \{0, 1, 2\}$, and $(d+j)$ mod $3 = i$.

The language $L(G)$ represented by this grammar is the set of all strings in Σ^* that represent integers divisible by three.

1. $L(G_2) \subseteq S_2$

   We have to show that every string w in $L(G_2)$ is an element of S_2. This can be proved by induction on the length of the derivation generating w.

   Induction basis. If w can be derived from the start symbol S of G_2 in one step, $S \Rightarrow_{G_2} w$, then $w = \lambda$ and, hence, $w \in S_2$.

   Next suppose $n > 1$. We assume, as *induction hypothesis*, that each string that can be derived from the start symbol S of G_2 by fewer than n replacement steps is an element of S_2. We need to show that each string that can be derived from S in G_2 by n steps is also an element of S_2.

   Let w be any arbitrary such string. Then there is a derivation

   $$S \Rightarrow_{G_2} w_1 \Rightarrow_{G_2} \cdots \Rightarrow_{G_2} w_n$$

   where $w = w_n$. Since $n > 1$ we must have $w_1 = (S)S$. In other words, the derivation is of the form

   $$S \Rightarrow_{G_2} (S)S \Rightarrow^*_{G_2} \cdots \Rightarrow^*_{G_2} (x)S \Rightarrow^*_{G_2} (x)y,$$

   where x and y are strings that can be derived from S in fewer than n steps. By the induction hypothesis, x and y are elements of S_2. By the definition of S, the string $w = (x)y$ is also an element of S_2.

2. $S_2 \subseteq L(G_2)$

   We have to show that every string w in S_2 can be derived from the start symbol S of G_2. This assertion can be proved by induction on the number of applications of recursion needed to produce w according to the definition of S_2.

   Induction basis. The only string $w \in S_2$ that can be obtained without any application of the recursive rule is the empty string λ, which can be derived from S in a single step, $S \Rightarrow_{G_2} \lambda$.

   Next suppose $n > 0$. We assume, as *induction hypothesis*, that any string in S_2 that can be obtained by fewer than n applications of the recursive rule can be derived from S in G_2. We need to show that each string in S_2 that can be obtained by n applications of the recursive rule can also be derived from S in G_2. Let w be any arbitrary such string.

   Since w requires at least one application of recursion, there exist strings x and y in S_2 such that $w = (x)y$ and x and y require fewer than n applications of the recursive rule. By the induction hypothesis, there are derivations

   $$S \Rightarrow^*_{G_2} x \text{ and } S \Rightarrow^*_{G_2} y.$$

   But then there is also a derivation

   $$S \Rightarrow_{G_2} (S)S \Rightarrow^*_{G_2} (x)S \Rightarrow^*_{G_2} (x)y,$$

   which shows that $w \in L(G_2)$.

Derivations

We have seen that the same string can possibly be generated in different ways, i.e., by different derivations, in a grammar.

For example, recall the grammar G_1 with rules

$$\begin{array}{lcl} S & \rightarrow & \lambda \\ S & \rightarrow & SS \\ S & \rightarrow & (S) \end{array}$$

and start symbol S.

The string ()() can be derived in five steps by

$$S \Rightarrow SS \Rightarrow (S)S \Rightarrow ()S \Rightarrow ()(S) \Rightarrow ()()$$

or

$$S \Rightarrow SS \Rightarrow S(S) \Rightarrow S() \Rightarrow (S)() \Rightarrow ()()$$

in G_1.

The string ()()() can be derived by

$$\begin{array}{l} S \Rightarrow SS \Rightarrow (S)S \Rightarrow ()SS \Rightarrow ()(S)S \Rightarrow ()()S \\ . \quad \Rightarrow ()()(S) \Rightarrow ()()() \end{array}$$

but also by

$$\begin{array}{l} S \Rightarrow SS \Rightarrow SSS \Rightarrow (S)SS \Rightarrow ()SS \Rightarrow ()(S)S \\ . \quad \Rightarrow ()()S \Rightarrow ()()(S) \Rightarrow ()()() \end{array}$$

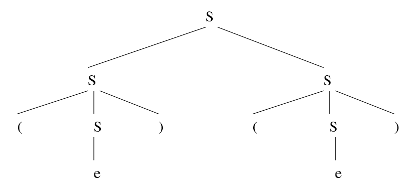
in G_1.

Parse Trees

Derivations are sequences of replacement steps but can also be represented in a more abstract way by labelled trees.

A *parse tree* for a grammar G is a labelled tree T, where (i) each leaf of T is labelled by an element of Σ or by the empty string λ, (ii) each interior (i.e., non-leaf) node of T is labelled by an element of V, and (iii) for each node i labelled by an element A of V there exists a rule $A \rightarrow x_1 x_2 \ldots x_n$, such that the children of i are labelled by $x_1, x_2, \ldots, x_n$ (in this order).

For example,



is a parse tree for the grammar G_1 (where e denotes λ).

The string one obtains by concatening the labels of leaves from left to right is called the *yield* of the tree.

Parse Trees (cont.)

The yield of the parse tree in the above example is the string ()().

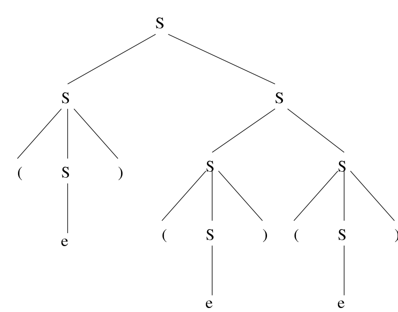
The tree can be constructed by starting with a single node labelled by S and then expanding it in several steps by adding each time children to a leaf according to one of the grammar rules.

In this sense the tree represents the derivations of ()() we had shown earlier. The constructions results in the same tree for both derivations because they employ the same replacement steps, only in a different order.

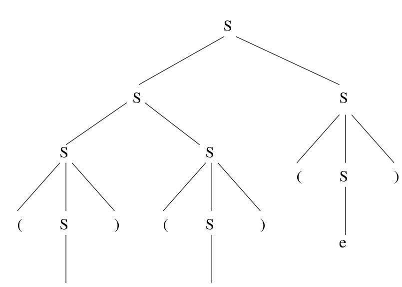
One can argue that the derivations are therefore *essentially the same* and that the tree representation captures their essence in a more abstract way.

Ambiguous Grammars

If we construct parse trees from the two derivations of ()()() we obtain two different trees,



and



This indicates that the grammar G_1 is *ambiguous*, as the structure of the string ()()() is not uniquely determined by the rules of G_1.

The grammar G_2, on the other hand, which defines the same language as G_1, is unambiguous and therefore preferable to G_1.

Example

Let L be the set of all strings in $\{a, b\}^*$ with an even number of a's.

Let R_1 be the set of rules

$$\begin{array}{lcl} S & \to & \lambda \\ S & \to & bS \\ S & \to & aA \\ A & \to & bA \\ A & \to & aS \end{array}$$

R_2 the set of rules

$$\begin{array}{lcl} S & \to & \lambda \\ S & \to & bS \\ S & \to & Sb \\ S & \to & aSa \end{array}$$

R_3 the set of rules

$$\begin{array}{lcl} S & \to & \lambda \\ S & \to & b \\ S & \to & aa \\ S & \to & aba \\ S & \to & SS \end{array}$$

R_4 the set of rules

$$\begin{array}{lcl} S & \to & \lambda \\ S & \to & AA \\ A & \to & AAA \\ A & \to & a \\ A & \to & bA \\ A & \to & Ab \end{array}$$

and R_5 the set of rules

$$\begin{array}{lcl} S & \to & \lambda \\ S & \to & bS \\ S & \to & SaSaS \end{array}$$

Let S be the start symbol in each case.

Which of these grammars generate L?

Answer. All except the grammar based on R_3. For example, the string $abba$ can not be derived from S by R_3.

We sketch next a proof that $L(R_1) = L$.

Lemma. For each string w in $(V \cup \Sigma)^*$, if $S \Rightarrow^*_{R_1} w$ then w contains an even number of a's, and if $A \Rightarrow^*_{R_1} w$ then w does not contain an even number of a's.

Proof sketch. By induction on the length of the derivation of w.

> We prove that for all $n > 0$, if w is a string in $(V \cup \Sigma)^*$, then
> (i) if $S \Rightarrow^n_{R_1} w$ then w contains an even number of a's and
> (ii) if $A \Rightarrow^n_{R_1} w$ then w does not contain an even number of a's.

Let n be an arbitrary, but fixed integer with $n > 0$.

Induction hypothesis. If $k < n$ and w is a string in $(V \cup \Sigma)^*$ then w contains an even number of a's, provided $S \Rightarrow^k_{R_1} w$, and w does not contain an even number of a's, provided $A \Rightarrow^k_{R_1} w$.

Induction step. We prove the above assertion for all strings w in $(V \cup \Sigma)^*$ for which $S \Rightarrow^n_{R_1} w$ or $A \Rightarrow^n_{R_1} w$, distinguishing several subcases depending on the first replacement step in the derivation.

Lemma. For each string w in Σ^*, if w contains an even number of a's then $S \Rightarrow^*_{R_1} w$, and if w does not contain an even number of a's then $A \Rightarrow^*_{R_1} w$.

Proof sketch. By induction on the length of the string w.

> We prove that for all $n > 0$, if w is a string in Σ^* and $|w| = n$, then
> (i) if w contains an even number of a's then $S \Rightarrow^*_{R_1} w$ and
> (ii) if w does not contain an even number of a's then $A \Rightarrow^*_{R_1} w$.

We consider two cases depending on whether w is the empty string or a non-empty string. If $w \neq \lambda$, then we further distinguish between two subcases, as either $w = a \cdot v$ or $w = b \cdot v$, for some string v.

Example - Non-Context-Free Grammar

Consider a grammar for a language over the alphabet $\Sigma = \{a\}$, with nonterminals $\{S, D, R, T, [,]\}$, start symbol S, and the following rules:

$$
\begin{array}{l}
\{S \rightarrow [Da],\\
S \rightarrow a,\\
Da \rightarrow aaD,\\
D] \rightarrow R],\\
D] \rightarrow T,\\
aR \rightarrow Ra,\\
[R \rightarrow [D,\\
aT \rightarrow Ta,\\
[T \rightarrow e\}
\end{array}
$$

This grammar, which is not context-free, generates the language $\{a^{2^n} : n \geq 0\}$.

The variable D is used to double the length of a string of a's:

$$Da^k \Rightarrow^* a^{2k}D.$$

The symbols [and] are used as left and right markers between which the generation of a string a^{2^k} takes place. The variable R initiates another application of doubling, whereas T is used to terminate the process.