Partial Orders

A binary relation on a set A is called a *partial order* if it is transitive and antisymmetric.

Examples of partial orders are the less-than relation (on the integers), the divisibility relation (on the integers), and the subset relation (on a power set).

If R is a partial order on A, one also speaks of a *partially ordered set* (A, R).

Note that a set, say the integers, can be partially ordered in different ways, e.g., by the less-than relation or the divisibility relation.

> *Is the empty set $\emptyset$ a partial order on a non-empty set A?*
>
> *Is the universal set $A \times A$ a partial order on A?*

We usually use infix notation for partial orders and write xRy to indicate that $(x, y) \in R$.

Two elements x and y are said to be *comparable* (with respect to R) if either xRy or yRx.

The symbols $\prec$ and $\preceq$ are often used to denote partial orders.

Quasi-Orders

We speak of a *reflexive* or *irreflexive* partial order, respectively, if the corresponding additional property is satisfied.

For example, the less-than relation on the integers is an irreflexive partial order, whereas the subset relation is a reflexive partial order.

Irreflexive partial orders are also called *quasi-orders*.

There is a natural correspondence between reflexive and irreflexive partial orders in that for every reflexive partial order one can define a corresponding quasi-order (by removing the "equality part" of the relation) and vice versa.

Lemma.

> Let E be the set $\{(x, x) : x \in A\}$.
>
> If R is a reflexive partial order on the set A, then $R \setminus E$ is an irreflexive partial order.
>
> If R is an irreflexive partial order on the set A, then the relation $R \cup E$ is a reflexive partial order.

Predecessors and Successors

Let $\prec$ be a partial order on a set A.

We say that x is a *predecessor* of y, and y a *successor* of x, if $x \neq y$ and $x \prec y$.

For example, the integer $k + j$ is a successor of k (with respect to the less-than relation) if j is positive, and a predecessor of k if j is negative.

The natural number 0 has no predecessor in the set of natural numbers (with respect to the less-than relation).

We say that x is an *immediate predecessor* of y, and y an *immediate successor* of x, iff $x \prec y$ and there is no element $z \in A$, such that x is a predecessor of z and z a predecessor of y.

The integer $k+1$ is an immediate successor of k, whereas $k - 1$ is an immediate predecessor (with respect to the less-than relation on the integers).

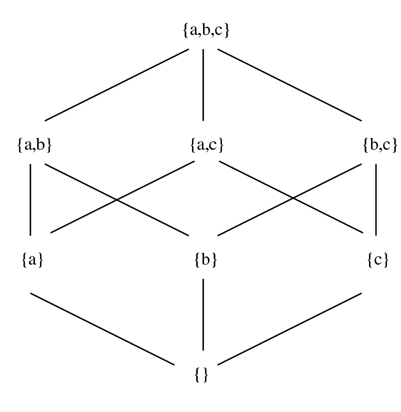
An element need not have an immediate predecessor or immediate successor. For instance, no rational number has an immediate predecessor or immediate successor with respect to the less-than relation (on $\mathbf{Q}$).

Hasse Diagrams

Partial orders on finite sets can be conveniently represented by graphs called "Hasse diagrams."

By the *Hasse diagram* of a (finite) partially ordered set $(A, \prec)$ we mean a directed graph with set of nodes A, the edges of which are all pairs (x, y) such that x is an immediate predecessor of y.

Hasse diagrams are usually drawn with their edges directed upwards (and with arrow heads left off).



Infinite Hasse Diagrams

It is possible to extend the presentation as Hasse diagrams to many infinite partial orders as well.

For example the set of integers with the less-than relation can be represented by an infinite Hasse diagram.

But "dense" orders such as the set of rational numbers with the less-than relation can not be represented in this way, as no rational number is an immediate predecessor of any other rational number.

Density refers to the property of an order that whenever an element x is a predecessor of an element y, then there is another element z such that x is a predecessor of z and z a predecessor of y.

Minima and Maxima

Let $(A, \prec)$ be a partially ordered set and S be a subset of A.

An element $x \in S$ is said to be *minimal* (in S with respect to $\prec$) if no element of S is a predecessor of x. If a minimal element x in S is a predecessor of each other element of S, then it is called a *least element* of S.

Similarly, an element $x \in S$ is said to be *maximal* if it has no successor in S. A maximal element that is a successor of each other element of S, is called a *greatest element* of S.

For example, the set $\{a, b, c\}$ is a greatest element in $\mathcal{P}(\{a, b, c\})$ with respect to the subset relation, whereas the empty set is a least element.

The sets $\{a\}$, $\{b\}$, and $\{c\}$ are minimal elements in the set of all non-empty subsets of $\{a, b, c\}$ (with respect to the subset relation).

The natural number 0 is the least element of $\mathbf{N}$ with respect to the less-than relation, but there are no maximal elements in this partially ordered set.

Note that maximal and minimal elements correspond to nodes at the top and bottom, respectively, in a Hasse diagram.

Thus, every finite partially ordered set has minimal and maximal elements, but infinite sets may have neither minimal nor maximal elements.

For example, the divisibility relation on the finite set $\{2, 4, 5, 10\}$ has two minimal elements, 2 and 5, and two maximal elements, 4 and 10. Consequently the set has no least element and no greatest element (with respect to the divisibility relation).

Lower and Upper Bounds

Let $\preceq$ be a reflexive partial order on a set A and S be a subset of A.

We say that x is a *lower bound of* S if $x \preceq y$ for each element y of S. (Note that x need not be an element of S.) Furthermore, if each other lower bound of S is a predecessor of x, then x is said to be a *greatest lower bound* of S.

A set may not have any minimal elements, yet still have lower bounds. For example, 0 is a lower bound of the set of positive rational numbers.

We say that x is an *upper bound of* S if $y \preceq x$ for each element y of S. If each other upper bound of S is a successor of x, then x is said to be a *least upper bound* of S.

For example, take the partially ordered set $(\mathbf{P}, |)$, where $|$ denotes the divisibility relation. Any common multiple of the two integers m and n is an upper bound of the two-element set $\{m, n\}$, whereas any common divisor of m and n is a lower bound of $\{m, n\}$.

Lattices

A *lattice* is a partially ordered set in which each subset $\{x,y\}$ has a least upper bound and a greatest lower bound.

Examples of lattices are the partially ordered sets $(\mathbf{N}, \leq)$ and $(\mathcal{P}(A), \subseteq)$.

More specifically, the greatest lower bound of a set of natural numbers $\{j,k\}$ is the minimum of the two numbers, whereas the least upper bound is the maximum of the two numbers.

In the case of subsets, the greatest lower bound of $\{X,Y\}$ is $X \cap Y$, whereas the least upper bound is $X \cup Y$.

Chains

Let $(A, \prec)$ be a partially ordered set.

A subset S of A is called a *chain* if any two distinct elements of S are comparable, i.e., if x and y are elements of S and $x \neq y$, then $x \prec y$ or $y \prec x$,

A (finite or infinite) sequence $x_1, x_2, \ldots$ of elements of A is called an *ascending chain* if $x_i \prec x_{i+1}$, for all $i \geq 1$. It is called a *descending chain* if $x_{i+1} \prec x_i$, for all $i \geq 1$.

For example, take the set of integers with the less-than relation. The sequence $1, 3, 5, 7, \ldots$ is an ascending chain, whereas $4, 2, 0, -2, -4, \ldots$ is a descending chain.

Linear Orders

A partial order $\prec$ is said to be *linear*, or *total*, if any two distinct elements x and y in A are comparable. i.e., either $x \prec y$ or $y \prec x$.

In other words, a partial order R on a set A is total if, and only if, A is a chain.

The less-than relation on the integers is linear, but the subset relation on a power set $\mathcal{P}(A)$ is not linear for most sets A.

> *For which sets A is the subset relation on $\mathcal{P}(A)$ linear?*

Product Orders

There are various ways in which partial orders can be extended to product sets (i.e., sets of tuples).

Let $(A, \prec_1)$ and $(B, \prec_2)$ be partially ordered sets. The corresponding *product order* $\prec_P$ is defined as follows:

$$(x,y) \prec_P (u,v) \text{ if and only if } x \prec_1 u \text{ and } y \prec_2 v.$$

For example, take $A = B = \mathbf{N}$ with the less-than-or-equal-to order. Then $(5,5) \leq_P (7,5)$, since $5 \leq 7$ and $5 \leq 5$. But $(5,2) \not\leq_P (100,1)$, because $2 \not\leq 1$.

Theorem.

> The product of two partial orders is a partial order.

Product orders can also be defined on n-fold cross products $A_1 \times A_2 \times \cdots \times A_n$:

$$(x_1, \ldots, x_n) \prec_P (y_1, \ldots, y_n) \text{ iff } x_i \prec_i y_i \text{ for } i = 1, \ldots, n$$

where $(A_i, \prec_i)$ is a partially ordered set, for $i = 1, \ldots, n$.

Filing Orders

Note that a product order need not be total, even when its component orders are total. But there is a different way of combining orders so that the combination of total orders will always result in a total order.

Let $(A_1, \prec_1), \ldots, (A_n, \prec_n)$ be partially ordered sets. We define a *filing relation* $\prec_F$ on the n-fold cross product $A_1 \times A_2 \times \cdots \times A_n$ by:

> $(x_1, \ldots, x_n) \prec_F (y_1, \ldots, y_n)$ iff
> there is an index k with $1 \leq k \leq n$ such that
> $x_1 = y_1, \ldots, x_{k-1} = y_{k-1}$ and $x_k \prec_k y_k$

For example, in this order $(5, 5) \prec_F (5, 7)$ and $(4, 100) \prec_F (5, 5)$.

Another instance of a filing order is the arrangement of personal files by last name (first) and first name (last).

Theorem.

1. The relations $\prec_F$ are partial orders.
2. A filing order is total if its component orders are total.

Filing orders are also called *lexicographic orders* on tuples.

Lexicographic Order on Strings

We next consider orders defined on sets of strings Σ^*, where Σ is an alphabet (i.e., a finite set).

Let $\prec$ be a partial order on Σ.

The *lexicographic relation* $\prec_L$ on Σ^* is defined by:

> $x \prec y$ if and only if either x is a proper prefix of y or else x and y have a longest common prefix u such that $x = uv$ and $y = uw$ and the first symbol of v is a predecessor of the first symbol of w (with respect to the partial order $\prec$ on Σ).

The arrangement of words in a dictionary is one example of a lexicographic order, and these orders are therefore also called *dictionary orders*.

For example, let Σ be the set $\{a, b\}$ with partial order $a \prec b$. Then $aa \prec_L ab$ and $aa \prec_L b$.

Theorem.

1. The relations $\prec_L$ are partial orders.
2. If $(\Sigma, \prec)$ is a totally ordered set, so is $(\Sigma^*, \prec_L)$.

In this order there are infinite ascending chains, e.g.,

$$\Lambda \prec_L a \prec_L aa \prec_L aaa \prec_L \cdots$$

as well as infinite descending chains, e.g.,

$$b \succ_L ab \succ_L aab \succ_L aaab \succ_L \ldots$$

Thus, the set of strings $\{a^n b : n \in \mathbf{N}\}$ has no minimal element.

Standard Order on Strings

Let Σ be an alphabet and $\prec$ be a partial order on Σ.

The *standard order* $\prec_S$ on Σ^* is defined by:

> $x \prec_S y$ if and only if either $|x| < |y|$ or else $|x| = |y|$ and $x \prec_L y$.

In other words, strings are compared first according to length and then lexicographically.

For example, let Σ be the set $\{a, b\}$ with partial order $a \prec b$. We have $aa \prec_S ab \prec_S ba \prec_S bb$. There are infinite ascending chains, e.g.,

$$\lambda \prec_S a \prec_S b \prec_S aa \prec_S \cdots bb \prec_S aaa \prec_S \cdots$$

but no infinite descending chains.

Theorem.

1. The relations $\prec_S$ are partial orders.
2. If $\prec$ is a total order on Σ, then $\prec_S$ is a total order on Σ^*.