Relations

A (binary) *relation* on sets A and B is a subset of the Cartesian product $A \times B$. In other words, the elements of a binary relation R are ordered pairs.

For example, if $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$, then $R = \{(a, 1), (b, 1), (b, 3)\}$ is a binary relation on A and B.

The sets A and B may be identical. For example, the less-than relation $<$ is a binary relation on **N** (and **N**).

Every function is a binary relation in this sense. Important examples of relations are (directed or undirected) graphs.

More generally, an *n-ary relation* on sets $A_1, \ldots, A_n$ is a subset of the n-fold Cartesian product $A_1 \times \cdots \times A_n$.

Reflexivity

Most of the relations we will discuss are characterized by some "internal" structure. In particular, we will study equivalence relations and order relations.

A binary relation R on a set A is said to be *reflexive* if $(x, x) \in R$, for all $x \in A$.

For example, the less-than relation is not reflexive, but the less-than-or-equal-to relation is.

For simplicity we often write "xRy" instead of "$(x, y) \in R$" when R is a binary relation.

A relation R on A is said to be *irreflexive* if xRx for *no* $x \in R$.

The less-than relation is irreflexive.

Note that irreflexivity is different from non-reflexivity. Every irreflexive relation R on a non-empty set A is also non-reflexive, but a non-reflexive relation need not be irreflexive.

Transitivity and Symmetry

A binary relation R on a set A is said to be *transitive* if whenever x, y, z are elements of A with xRy and yRz, then xRz.

Both the less-than relation and the less-than-or-equal-to relation are transitive.

An example of a non-transitive relation is the parent relation. The ancestor relation, though, is transitive.

A binary relation R on A is called *symmetric* if xRy implies yRx, for all x and y in A.

The equality relation is symmetric, but the less-than relation is not.

A relation R is called *antisymmetric* if xRy and yRx imply that x and y are identical, for all x and y.

The less-than relation is antisymmetric.

A relation may be neither symmetric nor antisymmetric.

Closure Properties

Let R and S be reflexive (or symmetric or transitive) relations on a set A. Do the union $R \cup S$ and intersection $R \cap S$ have the same property?

Let A be the set $\{1, 2, 3\}$ and consider binary relations

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$$
and
$$S = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}.$$

The relations R and S are reflexive, symmetric, and transitive.

The relation $R \cup S$ is reflexive and symmetric. It also contains the pairs $(1, 2)$ and $(2, 3)$, but not $(1, 3)$. Therefore it is not transitive.

In general, reflexivity and symmetry are preserved under set union, but as the example shows, transitivity is not always preserved.

In such cases one often considers extensions of a given relation that satisfy certain properties. Minimal such extensions are known as "closures."

We next discuss closures of relations under reflexivity, symmetry, and transitivity.

Reflexive Closures

Let R be a binary relation on a set A.

By the *reflexive closure* of R we mean the relation
$$r(R) = R \cup E,$$
where E denotes the set $\{(x,x) : x \in A\}$.

For example, if R is the $<$ relation on the integers, then $r(<)$ is the $\leq$ relation.

Theorem.

> If R is a binary relation on A, then $r(R)$ is a reflexive binary relation with $R \subseteq r(R)$.
>
> Furthermore, whenever S is a reflexive relation on A with $R \subseteq S$, then $r(R) \subseteq S$.

The first part of the theorem follows immediately from the definition of reflexive closure.

The second part states that $r(R)$ is the smallest reflexive relation (in a set-theoretic sense) that contains R as a subset.

Corollary.

> If R is reflexive, then $r(R) = R$.

Symmetric Closures

Let R be a binary relation on a set A.

By the *symmetric closure* of R we mean the relation
$$s(R) = R \cup R^c,$$
where R^c denotes the converse of R, i.e., the set $\{(y,x) : (x,y) \in R\}$.

For example, if R is the relation $\{(1,1),(1,2),(1,3)\}$, then

> $R^c = \{(1,1),(2,1),(3,1)\}$
> and $s(R) = \{(1,1),(1,2),(2,1),(1,3),(3,1)\}$.

Theorem.

> If R is a binary relation on A, then $s(R)$ is a symmetric binary relation with $R \subseteq s(R)$.
>
> Furthermore, whenever S is a symmetric relation on A with $R \subseteq S$, then $s(R) \subseteq S$.

The first part of the theorem again follows immediately from the definition of symmetric closure, and the second part states that $s(R)$ is the smallest symmetric relation that contains R as a subset.

Corollary.

> If R is symmetric, then $s(R) = R$.

Composite Relations

Let R be a relation on $A \times B$ and S a relation on $B \times C$. Then the *composite relation* $R \circ S$, or simply RS, is defined to be the set

> $\{(x,z) \in A \times C : \text{ for some } y \in B,\ xRy \text{ and } ySz\}$.

For example, if $R = \{(1,a),(2,a),(3,b),(4,c)\}$ and $S = \{(a,b),(b,a)\}$, then $RS = \{(1,b),(2,b),(3,a)\}$.

If R is a binary relation on A, i.e., a subset of $A \times A$, we define the "k-fold" composition of R by induction as follows.

$$R^k = \begin{cases} R & \text{if } k = 1 \\ R^{k-1}R & \text{if } k > 1 \end{cases}$$

Lemma.

> We have $R^j \circ R^k = R^{j+k}$, for all $j,k \geq 1$.

Transitive Closures

Let R be a binary relation on a set A.

By the *transitive closure* of R we mean the relation
$$t(R) = \bigcup_{k \geq 1} R^k.$$

For example, if R is the relation $\{(1,1),(1,2),(2,3)\}$, then $t(R) = \{(1,1),(1,2),(2,3),(1,3)\}$.

Theorem.

> If R is a binary relation on A, then $t(R)$ is a transitive binary relation with $R \subseteq t(R)$.
>
> Furthermore, whenever S is a transitive relation on A with $R \subseteq S$, then $t(R) \subseteq S$.

The proof of this theorem is not as straightforward as the proofs of the corresponding theorems for reflexivity and symmetry. The second part states that $t(R)$ is the smallest transitive relation that contains R as a subset.

Corollary.

> If R is transitive, then $t(R) = R$.

Properties of Closures

Proposition

> If R is a binary relation on A, then $r(r(R)) = r(R)$, $s(s(R)) = s(R)$, and $t(t(R)) = t(R)$.

Theorem

1. If R is reflexive, then so are $s(R)$ and $t(R)$.

2. If R is symmetric, then so are $r(R)$ and $t(R)$.

3. If R is transitive, then so is $r(R)$.

This theorem implies the validity of various identities between relations. For instance, $r(s(r(R))) = s(r(R))$ or $s(r(s(R))) = s(r(R))$.

Exercise. Show that $r(s(R)) = s(r(R))$, for all binary relations R on a set A.

An important consequence of the previous results is:

Theorem

> If R is a binary relation on a set A, then $t(r(s(R)))$ is a reflexive, symmetric, and transitive relation on A.

Does $s(t(r(R)))$ also satisfy these three properties?