Well-Founded Orders

A partially ordered set $(A, \prec)$ is said to be *well-founded* if every descending sequence of elements is finite, i.e., there are no infinite descending sequences.

For example, the set of natural numbers with the less-than relation is well-founded. But the set of integers with the same relation is not well-founded, e.g.,

$$0, -1, -2, -3, \ldots$$

Theorem.

If $(A, \prec)$ is well-founded, then each non-empty subset of A has a minimal element (with respect to $\prec$).

Conversely, if every non-empty subset of A has a minimal element then $(A, \prec)$ is well-founded.

Note that if a well-founded set if totally ordered, then every nonempty subset has a least element.

Other examples of well-founded sets include Cartesian products $A \times B$ with product orders based on well-founded component orders; and sets of strings Σ^* with the standard order.

Well-Founded Induction

Well-founded orders are the basis for the following general mathematical induction principle:

Principle of Well-Founded Induction

Let $\prec$ be a well-founded ordering on a set S and P be a property defined on elements of S.

If

for all elements x in S, whenever $P(y)$ is true for all predecessors y of x, then $P(x)$ is also true,

then

$P(x)$ is true for all x in S.

This principle can be used to (a) prove properties on well-founded sets and (b) define properties (or functions) on well-founded sets.

Example of Well-Founded Induction

We show how to use well-founded induction to prove that

every natural number greater than 1 *is divisible by a prime number.*

Let S be the set of positive integers and $\prec$ be the usual less-than relation.

Proof.

Let $P(x)$ be the property that x is divisible by a prime number.

Suppose x is a natural number with $1 < x$. Let us assume (as "induction hypothesis") that $P(y)$ is true for all predecessors of x in S, i.e., for all natural numbers y such that $1 < y < x$. We must show that $P(x)$ is true.

We distinguish between two cases.

(a) If x is a prime number then $P(x)$ is evidently true as x divides itself.

(b) If x is not a prime number then x can be written as a product, $x = y * z$, where $1 < y, z < x$. By the induction hypothesis, $P(y)$ is true and hence y is divisible by a prime. But since y divides x this implies that x is also divisible by a prime number.

Ackermann's Function

Ackermann's function is defined as follows:

$$\begin{aligned} f(x,y) \quad = \quad & \text{if } x = 0 \text{ then } y + 1 \\ & \text{else if } y = 0 \text{ then } f(x-1, 1) \\ & \text{else } f(x-1, f(x, y-1)) \end{aligned}$$

One can use well-founded induction with respect to the filing order on $\mathbf{N} \times \mathbf{N}$ to show that this function is defined for all pairs of natural numbers.

Standard Mathematical Induction

An important special case of well-founded induction is its application to the natural numbers with the usual less-than order.

Standard Mathematical Induction

> Let $P(n)$ be a predicate defined on the natural numbers and let a be a fixed natural number. Suppose the following two statements are true:
>
> 1. $P(a)$ is true.
> 2. For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.
>
> Then $P(n)$ is true for all natural numbers n with $n \geq a$.

Usually, the constant a is chosen to be 0.

Standard Induction Proofs

Standard induction proofs consist of two steps:

1. the *basis step*: proving that $P(a)$ is true; and
2. the *inductive step*: proving that for an arbitrary, but fixed number $n \geq a$, $P(n+1)$ is true whenever $P(n)$ is true.

For example, let $P(n)$ be the property "$2^n > n^3$" and let us prove that $P(n)$ is true for all $n \geq 10$.

Basis step. If $n = 10$, then

$$2^n = 2^{10} = 1024 > 1000 = 10^3 = n^3.$$

Inductive step. Assume $P(n)$ is true for some arbitrary, but fixed integer $n \geq 10$. We have to show that $P(n+1)$ is true.

$$\begin{array}{lll} 2^{n+1} & = & 2 * 2^n \\ & > & 2 * n^3 \quad \text{(by inductive hypothesis)} \\ & = & n^3 + n^3 \\ & > & n^3 + 7n^2 \quad \text{(because } n \geq 10\text{)} \\ & > & n^3 + 3n^2 + 3n + 1 \quad \text{(because } n \geq 10\text{)} \\ & = & (n+1)^3 \quad \text{(by basic algebra)} \end{array}$$

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Note that the assumption that $P(n)$ is true is called the *inductive hypothesis*. It is part of the inductive step.

Example - Fibonacci Numbers

Recall that the Fibonacci numbers are defined recursively:

$$F_n = \begin{cases} 1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n > 1 \end{cases}$$

Let ϕ be the number $(1 + \sqrt{5})/2$.

We claim that

$$F_n \leq \phi^n$$

for all natural numbers n.

We first prove *two* basis cases.

If $n = 0$, then

$$F_n = F_0 = 1 = \phi^0 = \phi^n.$$

If $n = 1$, then

$$F_n = F_1 = 1 < 3/2 < (1 + \sqrt{5})/2 = \phi = \phi^n.$$

Next let us assume, for some arbitrary, but fixed positive integer n, that

$$F_{n-1} \leq \phi^{n-1} \text{ and } F_n \leq \phi^n.$$

We show that under these assumptions

$$F_{n+1} \leq \phi^{n+1}.$$

More specifically we have

$$\begin{array}{lll} F_{n+1} & = & F_n + F_{n-1} \quad \text{(by def. of } F_{n+1}\text{)} \\ & \leq & \phi^n + \phi^{n-1} \quad \text{(by assumptions)} \\ & = & \phi^{n-1}(\phi + 1) \\ & = & \phi^{n-1}\phi^2 \quad \text{(see below)} \\ & = & \phi^{n+1} \end{array}$$

Note that

$$\begin{array}{lllll} \phi^2 & = & (6 + 2\sqrt{5})/4 & = & (3 + \sqrt{5})/2 \\ & = & 1 + (1 + \sqrt{5})/2 & = & 1 + \phi. \end{array}$$

Another Example

Let $P(n)$ be the property that

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6.$$

We use induction to prove that $P(n)$ is true for all integers $n \geq 1$.

Basis step. If $n = 1$, then

$$\sum_{i=1}^{n} i^2 = 1^2 = 1 = (1 \cdot 2 \cdot 3)/6 = n(n+1)(2n+1)/6.$$

Inductive step. Assume as induction hypothesis that $P(n)$ is true for some arbitrary, but fixed integer $n \geq 1$. i.e.,

$$\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6.$$

We have to show that $P(n+1)$ is also true, or equivalently

$$\sum_{i=1}^{n+1} i^2 = (n+1)(n+2)(2n+3)/6.$$

We have

$$\begin{aligned}
\textstyle\sum_{i=1}^{n+1} i^2 &= (\textstyle\sum_{i=1}^{n} i^2) + (n+1)^2 \\
&= n(n+1)(2n+1)/6 + (n+1)^2 \quad (by I.H.) \\
&= (n+1)(2n^2+n)/6 + (n+1)(n+1)6/6 \\
&= (n+1)[(2n^2+n) + 6(n+1)]/6 \\
&= (n+1)(2n^2+7n+6)/6 \\
&= (n+1)(n+2)(2n+3)/6
\end{aligned}$$

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Inductively Defined Structures

Many data structures are formally defined by mathematical induction. We have seen several examples, for instance, lists of elements of a certain type t.

Basis.

> The empty list, denoted by [] or nil, is a t-list.

Induction.

> If L is a t-list and a is an object of type t, then $a :: L$ is a t-list.

The key observation is that such a definition uniquely specifies a set of data objects and that the inherent inductive structure can be used to prove properties about the data type or functions defined on it.

Another example is the set of all strings that can be composed from characters of a given set Σ. This set Σ^* can be defined inductively as follows:

Basis.

> The empty string, denoted by ϵ, is an element of Σ^*.

Induction.

> If a is a symbol in Σ and s is a string in Σ^*, then $a \cdot s$ is also a string in Σ^*.

Constructing Well-Founded Orders

Partial orders on finite sets are well-founded.

On an infinite set A a well-founded order can be constructed by specifying a mapping from A to a known well-founded set, such as the natural numbers.

More specifically, let f be a function from A to $\mathbf{N}$. Define an order $\prec_f$ by:

> $x \prec_f y$ if and only if $f(x) < f(y)$.

Lemma.

> The relations $\prec_f$ are well-founded partial orders on A.

If the set A has been defined inductively there is a natural way of defining a function $f : A \to \mathbf{N}$ as follows:

1. If x is a basis element of A, then $f(x) = 0$.

2. If x is constructed from elements $y_1, \ldots, y_n$, then $f(x) = 1 + \max(f(y_1), \ldots, f(y_n))$.

For instance, in this way we obtain for the (inductively defined) set Σ^* a well-founded order in which strings are compared according to their length.