

CSE328 Fundamentals of Computer Graphics: Theory, Algorithms, and Applications

Hong Qin

Department of Computer Science

State University of New York at Stony Brook (Stony
Brook University)

Stony Brook, New York 11794-4400

Tel: (631)632-8450; Fax: (631)632-8334

qin@cs.sunysb.edu

<http://www.cs.sunysb.edu/~qin>

2D-3D Transformations

- From local, model coordinates to global, world coordinates

Modeling Transformations

- 2D-3D transformations
- Specify transformations for objects
 - Allows definitions of objects in their own coordinate systems
 - Allows use of object definition multiple times in a scene
 - Please pay attention to how OpenGL provides a transformation stack because they are so frequently reused

Overview

- **2D Transformations**
 - Basic 2D transformations
 - Matrix representation
 - Matrix composition
- **3D Transformations**
 - Basic 3D transformations
 - Same as 2D

Basic 2D Transformations

- **Translation:**

- $x' = x + t_x$

- $y' = y + t_y$

- **Scale:**

- $x' = x * s_x$

- $y' = y * s_y$

- **Shear:**

- $x' = x + h_x * y$

- $y' = y + h_y * x$

- **Rotation:**

- $x' = x * \cos\Theta - y * \sin\Theta$

- $y' = x * \sin\Theta + y * \cos\Theta$

Scaling

- **Scaling** a coordinate means multiplying each of its components by a scalar
- **Uniform scaling** means this scalar is the same for all components:
- **Non-uniform scaling**: different scalars per component:
- **How can we represent scaling in matrix form?**

Scaling Operation in Matrix Form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Scaling

- **Scaling operation:**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$$

- **Or, in matrix form:**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

scaling matrix

Rotation

2-D Rotation

$$x' = x \mathbf{\cos}(\theta) - y \mathbf{\sin}(\theta)$$

$$y' = x \mathbf{\sin}(\theta) + y \mathbf{\cos}(\theta)$$

2D Rotation Derivation

- $x = r \cos(\phi)$
- $y = r \sin(\phi)$
- $x' = r \cos(\phi + \theta)$
- $y' = r \sin(\phi + \theta)$

- $x' = r \cos(\phi) \cos(\theta) - r \sin(\phi) \sin(\theta)$
- $y' = r \sin(\phi) \sin(\theta) + r \cos(\phi) \cos(\theta)$

- $x' = x \cos(\theta) - y \sin(\theta)$
- $y' = x \sin(\theta) + y \cos(\theta)$

2-D Rotation

- It is straightforward to see this procedure in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Even though $\sin(\theta)$ and $\cos(\theta)$ are nonlinear functions of θ ,

- x' is a linear combination of x and y
- y' is a linear combination of x and y

2D Rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Basic 2D Transformations

- **Translation:**

- $x' = x + t_x$

- $y' = y + t_y$

- **Scale:**

- $x' = x * s_x$

- $y' = y * s_y$

- **Shear:**

- $x' = x + h_x * y$

- $y' = y + h_y * x$

- **Rotation:**

- $x' = x * \cos\Theta - y * \sin\Theta$

- $y' = x * \sin\Theta + y * \cos\Theta$

Basic 2D Transformations

- **Translation:**

- $x' = x + t_x$
- $y' = y + t_y$

- **Scale:**

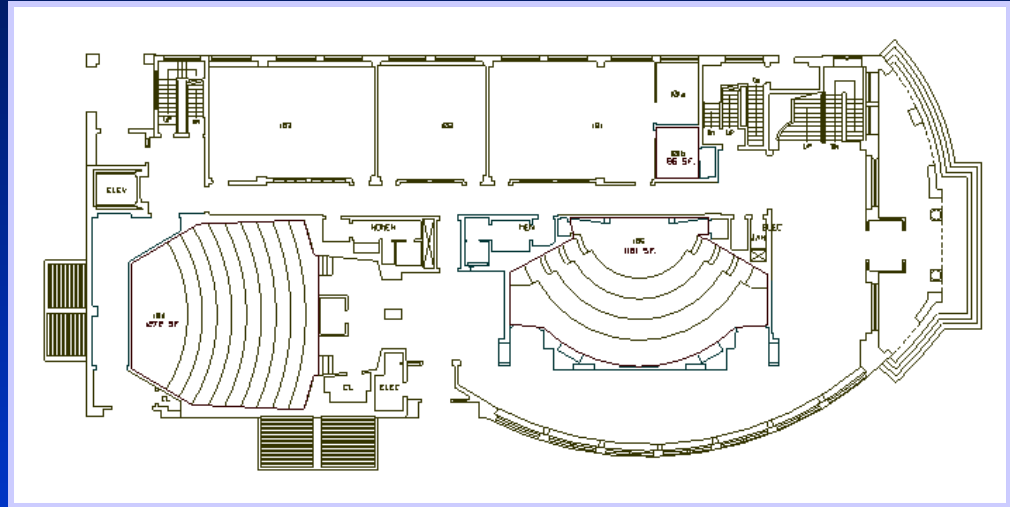
- $x' = x * S_x$
- $y' = y * S_y$

- **Shear:**

- $x' = x + h_x * y$
- $y' = y + h_y * x$

- **Rotation:**

- $x' = x * \cos\theta - y * \sin\theta$
- $y' = x * \sin\theta + y * \cos\theta$



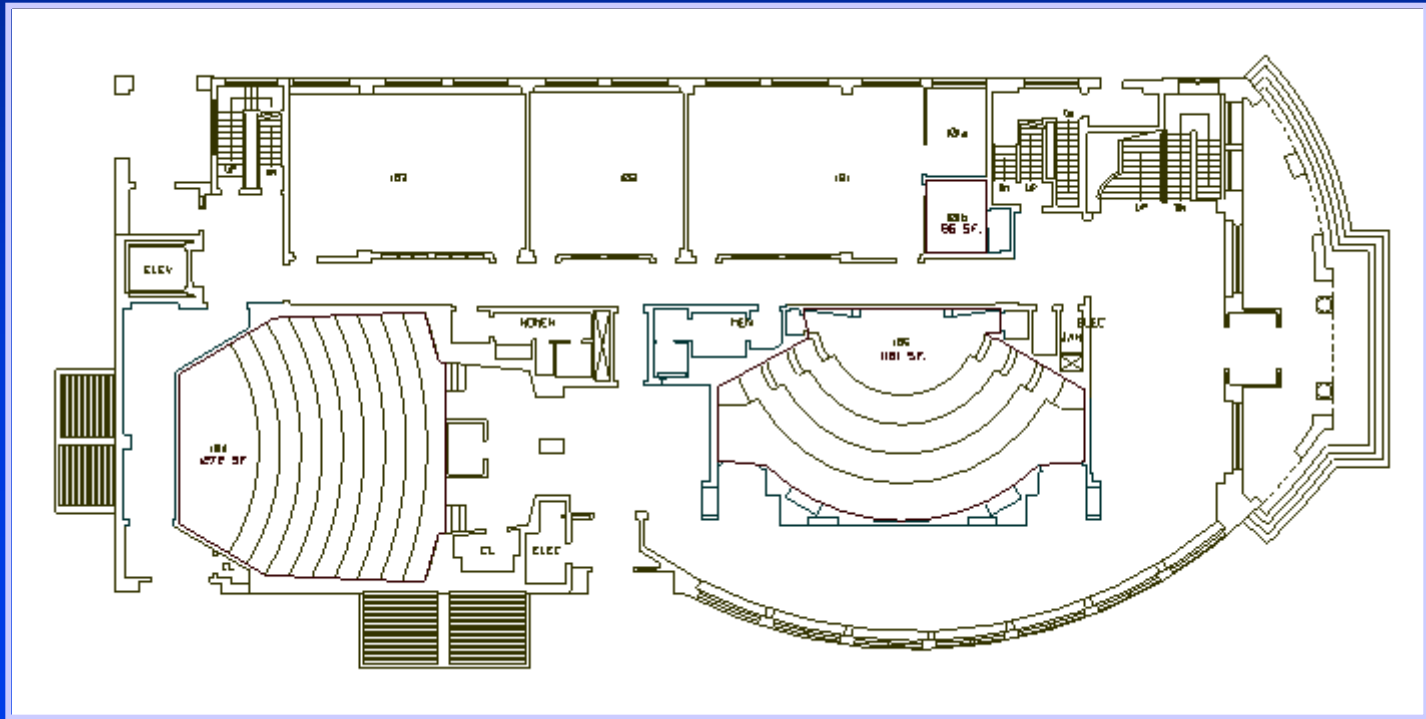
Transformations can be combined (with simple algebra)

Combining Transformations

- Transformations can be combined (with simple algebra)

Composite Transformations

Transformations can be combined
(with simple algebra)



Matrix Representation

- Represent 2D transformation by a matrix
- Multiply matrix by column vector
 \Leftrightarrow apply transformation to point

Overview

- **2D Transformations**
 - Basic 2D transformations
 - Matrix representation
 - Matrix composition
- **3D Transformations**
 - Basic 3D transformations
 - Same as 2D

Matrix Representation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

Matrix Representation

- Represent 2D transformation by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Multiply matrix by column vector
 \Leftrightarrow apply transformation to point

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

Matrix Representation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix Representation

- Transformations can be combined by multiplication
 - Matrices are a convenient and efficient way to represent a sequence of transformations!

Matrix Representation

- Transformations combined by multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrices are a convenient and efficient way to represent a sequence of transformations!

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Identity?

$$\begin{aligned}x' &= x \\y' &= y\end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Scale around (0,0)?

$$\begin{aligned}x' &= s_x * x \\y' &= s_y * y\end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Rotate around (0,0)?

$$\begin{aligned}x' &= \cos\Theta * x - \sin\Theta * y \\y' &= \sin\Theta * x + \cos\Theta * y\end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Shear?

$$\begin{aligned}x' &= x + sh_x * y \\y' &= sh_y * x + y\end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & sh_x \\ sh_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Mirror about Y axis?

$$\begin{aligned}x' &= -x \\ y' &= y\end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Mirror over (0,0)?

$$\begin{aligned}x' &= -x \\ y' &= -y\end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Translation?

$$x' = x + t_x$$

$$y' = y + t_y$$

NO!

Only linear 2D transformations
can be represented with a 2x2 matrix

Linear Transformations

- Linear transformations are combinations of ...

- Scale,
- Rotation,
- Shear, and
- Mirror

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Properties of linear transformations:

- Satisfies:
- Origin maps to origin
- Lines map to lines
- Parallel lines remain parallel
- Ratios are preserved
- Closed under composition

$$T(s_1\mathbf{p}_1 + s_2\mathbf{p}_2) = s_1T(\mathbf{p}_1) + s_2T(\mathbf{p}_2)$$

Homogeneous Coordinates

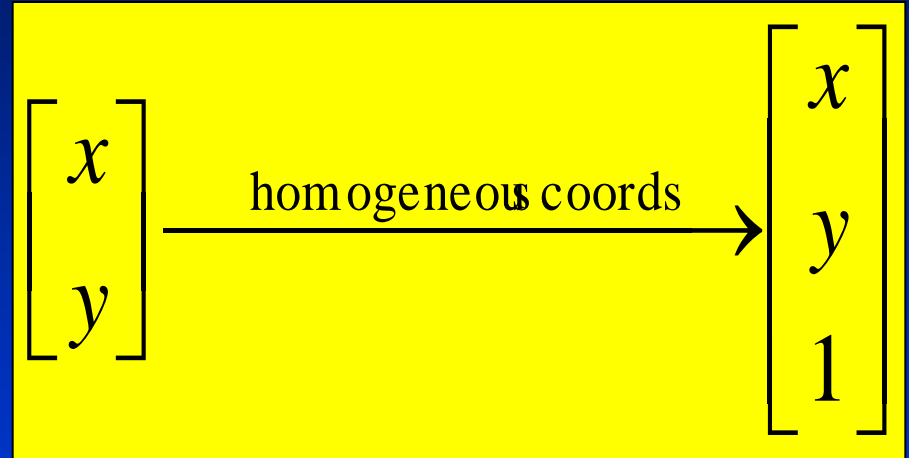
- Q: How can we represent translation as a 3x3 matrix?

$$x' = x + t_x$$

$$y' = y + t_y$$

Homogeneous Coordinates

- **Homogeneous coordinates**
 - represent coordinates in 2 dimensions with a 3-vector



Homogeneous coordinates appear to be far less intuitive, but they indeed make graphics operations much easier

Homogeneous Coordinates

- Q: How can we represent translation as a 3x3 matrix?

$$\begin{aligned}x' &= x + t_x \\ y' &= y + t_y\end{aligned}$$

- A: Using the rightmost column:

$$\text{Translation} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Translation

- Example of translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

- Add a 3rd coordinate to every 2D point
 - (x, y, w) represents a point at location $(x/w, y/w)$
 - $(x, y, 0)$ represents a point at infinity
 - $(0, 0, 0)$ is not allowed
- Note that, $(6,3,1)$; $(12,6,2)$; and $(18,9,3)$ represent the **SAME POINT** in 2D

Convenient coordinate system to represent many useful transformations

Basic 2D Transformations

- Basic 2D transformations as 3x3 matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & sh_x & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

Affine Transformations

- Affine transformations are combinations of ...

- Linear transformations, and
- Translations

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties of affine transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- Parallel lines remain parallel
- Ratios are preserved
- Closed under composition

Projective Transformations

- Projective transformations ...

- Affine transformations, and
- Projective warps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties of projective transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- Parallel lines do not necessarily remain parallel
- Ratios are not preserved
- Closed under composition

Overview

- **2D Transformations**
 - Basic 2D transformations
 - Matrix representation
 - Matrix composition
- **3D Transformations**
 - Basic 3D transformations
 - Same as 2D

Matrix Composition

- Transformations can be combined by matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$$\mathbf{p}' = T(t_x, t_y) * R(\Theta) * S(s_x, s_y) * \mathbf{p}$$

Matrix Composition

- **Matrices are a convenient and efficient way to represent a sequence of transformations**
 - General purpose representation
 - Hardware matrix multiply

$$\mathbf{p}' = (T * (R * (S * \mathbf{p})))$$

$$\mathbf{p}' = (T * R * S) * \mathbf{p}$$

Matrix Composition

- From local coordinates to global coordinates
- Be aware: order of transformations matters
 - Matrix multiplication is not commutative

$$\mathbf{p}' = \mathbf{T} * \mathbf{R} * \mathbf{S} * \mathbf{p}$$



Matrix Composition

- A more complicated example: rotating 90 degrees around the mid-point of a line segment (whose coordinates are $(3,2)$)
- Can we change the order between rotation and translation?

Matrix Composition

Will this sequence of operations work?

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(90) & -\sin(90) & 0 \\ \sin(90) & \cos(90) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ 1 \end{bmatrix} = \begin{bmatrix} a'_x \\ a'_y \\ 1 \end{bmatrix}$$

Matrix Composition

- After correctly ordering the matrices
- Multiply matrices together
- What results is one matrix – store it (on stack)!
- Multiply this matrix by the vector of each vertex
- All vertices easily transformed with one matrix multiply

Overview

- **2D Transformations**
 - Basic 2D transformations
 - Matrix representation
 - Matrix composition
- **3D Transformations**
 - Basic 3D transformations
 - Same as 2D

3D Transformations

- Same idea as 2D transformations
 - Homogeneous coordinates: (x, y, z, w)
 - 4x4 transformation matrices

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Basic 3D Transformations

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Identity

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Mirror about Y/Z plane

Basic 3D Transformations

Rotate around Z axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 & 0 \\ \sin\Theta & \cos\Theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Rotate around Y axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} \cos\Theta & 0 & \sin\Theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\Theta & 0 & \cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Rotate around X axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Theta & -\sin\Theta & 0 \\ 0 & \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Reverse Rotations

- Q: How do you undo a rotation of θ , $R(\theta)$?
- A: Apply the inverse of the rotation.... $R^{-1}(\theta) = R(-\theta)$
- How to construct $R^{-1}(\theta) = R(-\theta)$
 - Inside the rotation matrix: $\cos(\theta) = \cos(-\theta)$
 - The cosine elements of the inverse rotation matrix are unchanged
 - The sign of the sine elements will flip
 - This is because the rotation matrix is orthogonal matrix
- Therefore... $R^{-1}(\theta) = R(-\theta) = R^T(\theta)$

Summary

- **Coordinate systems are the basis for computer graphics**
 - World vs. model coordinates (Global vs. Local)
- **2-D and 3-D transformations**
 - Trigonometry and geometry
 - Matrix representations
 - Linear, affine, and projective transformations
- **Matrix operations**
 - Matrix composition