



The Regular Operations

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- This requires **appropriate tools** and **technique** which we initiate here
- We also need **tools** and **techniques** for **studying non-regular languages**, i.e., languages which are beyond the capability of finite automata

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- **Arithmetic:** basic objects are numbers and the tools are operations for number manipulation, such as $+$, \times .
- **Theory of Computation:** objects are languages and the tools for language manipulation, specifically designed
- The three common operations on languages, regular operations: union \cup , concatenation \circ , and star \star .

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- **Concatenation:** $A \circ B = \{xy \mid x \in A \wedge y \in B\}$
- **Star:** $A^* = \{x_1x_2 \dots x_k \mid k \geq 0 \wedge x_i \in A, 1 \leq i \leq k\}$

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- What are these alphabets?
 - Answer: if Σ_A and Σ_B are the alphabets of A and B then these alphabets are $\Sigma_A \cup \Sigma_B$
 - Reason: Any language over Σ_A or Σ_B is certainly a language over $\Sigma_A \cup \Sigma_B$. Hence, we may assume $\Sigma_A = \Sigma_B$

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- **Concatenation** is a little trickier; it attaches a string from A in front of a string from B in all possible ways to get strings from $A \circ B$
- **Star** operation is different; it applies to one language, i.e., it is unary rather than binary. Star works by attaching any number of strings in A together to get strings in A^*

Note

Because “any number” includes 0 , $\epsilon \in A^*$,
no matter what A is.

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- $A \cup B = \{good, bad, boy, girl\}$
- $A \circ B = \{goodboy, goodgirl, badboy, badgirl\}$
- $A^* = \{\epsilon, goob, bad, goodgood, goodbad, badgood, badbad, goodgoodgood, goodgoodbad, goodbadgood, goodbadbad, \dots\}$

Closed set

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- Since the division of two natural numbers is not always a natural number, for example $1/2 \notin \mathcal{N}$, \mathcal{N} is not closed under division

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- This property provides useful tools for manipulating regular languages and for understanding the power of finite automata

Theorem 1.24

The class of regular languages is closed under union operation, i.e if A_1 and A_2 are regular then $A_1 \cup A_2$ is regular.

Proof idea

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- The machine M works by simulating M_1 and M_2
- **Simulation:** pretend that you are M . As you read the input symbols you simulate both M_1 and M_2 , simultaneously
- To keep track of both simulations, need to remember the state each machine would be in if it had read up to this point in the input

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- Transitions of M goes from pair to pair, updating the state for both M_1 and M_2
- The start state of M is the pair of start states of M_1 and M_2 ; the accept states of M is the set of pairs containing an accept state of M_1 or M_2

Proof

By construction. Let M_1 recognize A_1 where
 $M_1 = (Q_1, \Sigma, \delta_1, q_0^1, F_1)$, and M_2 recognize A_2 where
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Construct M to recognize $A_1 \cup A_2$,

$M = (Q, \Sigma, \delta, q_0, F)$, where:

Construction of M

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- For each $(r_1, r_2) \in Q$ and $a \in \Sigma$,
$$\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$$

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- For each $(r_1, r_2) \in Q$ and $a \in \Sigma$,
 $\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$
- $q_0 = (q_0^1, q_0^2)$
- $F = \{(r_1, r_2) \mid r_1 \in F_1 \vee r_2 \in F_2\}$, i.e., $F = (F_1 \times Q_2) \cup (Q_1 \times F_2)$.
Note, this is not the same as $F_1 \times F_2$

Corollary

Class of regular languages is closed under intersection.

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Proof: For two regular languages A and B , recognized by the automata M_A and M_B the automaton that recognizes the language $A \cap B$ is constructed in the same way as the automaton that recognizes the language $A \cup B$ with the final states defined by $F = \{(r_1, r_2) \mid (r_1, r_2) \in F_1 \times F_2\}$

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- More complicated constructions require additional discussion to prove correctness.
- A formal correctness proof for a construction of this type usually proceeds by induction. We will illustrate it further

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The class of regular languages is closed under concatenation operation, i.e if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

Proof idea: As before, we can start with finite automata M_1 and M_2 recognizing A_1 and A_2 and construct the automaton M to recognize $A_1 \circ A_2$

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Ideas for construction of M

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- The problem is that M does not know where to break its input.
- To solve this problem we need to introduce a new technique: *the nondeterminism*