CSE371 FINAL Fall 2011
(100pts + 15 extra credit pts

NAME **ID:** **Math/CS**

There are 5 problems. Each is 20 pts.
Problem 6 is 15 extra poinrs

QUESTION 1

1. For the sentence

If it is not true that: $2 > 6$ *and today is Monday, then the fact that* $2 > 6$ *implies that today is Monday.*

write its corresponding formula A. Explain your solution.

2. Define a formal language to which the formula A belongs.

3. List all proper, non-atomic sub-formulas of A

4. Find a model (restricted) and a restricted counter-model for A (classical semantics). Use short-hand notation. Show work.

5. List 3 models and 3 counter-models for A by extending the restricted model and the counter-model you have found in 4. to the set VAR of all variables.

QUESTION 2 The algebraic models for the intuitionistic logic are defined in terms of *Pseudo-Boolean Algebras* in the following way.

A formula A is said to be an intuitionistic tautology if and only if $v \models A$, for all v and all Pseudo-Boolean Algebras, where v maps VAR into universe of a Pseudo-Boolean Algebra.

I.e. A is an intuitionistic tautology if and only if it is true in all Pseudo-Boolean Algebras under all possible variable assignments.

The 3 element Heyting algebra $H3$ defined below is a 3 element Pseudo-Boolean Algebra.

$$H3 = (\{F, \bot, T\}, \cup, \cap, \Rightarrow, \neg)$$

for the operations $\neg, \Rightarrow, \cup, \cap, \neg$ of **H3** are defined as follows. (We assume that $\{F < \bot < T\}$.)

For any $a, b \in \{F, \bot, T\}$,

$$a \cup b = max\{a, b\},$$

$$a \cap b = min\{a, b\}.$$

$$a \Rightarrow b = \begin{cases} T & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$

$$\neg a = a \Rightarrow F.$$

1. Verify whether the 3- element Heyting algebra is a model for the following formula, i.e if it is true that

$$\models_{H3} ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)).$$

2. Use the the 3- element Heyting algebra to determine whether the formulas below

$$((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)),$$

is not Intuitionistic Logic tautology.

QUESTION 3 H is the following proof system:

$$H = (\ \mathcal{L}_{\{\Rightarrow, \neg\}},\ \mathcal{F},\ AX = \{A1, A2, A3, A4, A5\},\quad MP\ )$$

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

A4 $(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$

A5 $(((A \Rightarrow B) \Rightarrow A) \Rightarrow \neg A)$

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

(1) Does Deduction Theorem holds for H? Explain.

(2) Is H COMPLETE with respect to all classical semantics tautologies? Explain

(3) Prove the following: $A \vdash_H (A \Rightarrow A)$

(4) We know that $\vdash_H (\neg A \Rightarrow (A \Rightarrow B))$. Prove, that $\neg A, A \vdash_H B$.

QUESTION 4 Let **GL** be the sound Gentzen style proof system for classical logic defined in chapter 11.

1. Prove, by constructing a proper decomposition tree that

$$\vdash_{\mathbf{GL}}((\neg(a \cap b) \Rightarrow b) \Rightarrow (\neg b \Rightarrow (\neg a \cup \neg b))).$$

2. Show that $\nvdash_{\mathbf{GL}} ((\neg a \Rightarrow b) \Rightarrow (b \Rightarrow \neg a))$.

3. Prove by construction of a a counter-model determined by a decomposition tree defined in **2.** that

$\not\models ((\neg a \Rightarrow b) \Rightarrow (b \Rightarrow \neg a))$

QUESTION 5 We know that a classical tautology $(\neg(a\cap b)\cup(a\cap b))$ is NOT Intuitionistic tautology.

1. Show, that the formula

$$\neg\neg(\neg(a \cap b) \cup (a \cap b))$$

is PROVABLE in the Gentzen system **LI** for Intuitionistic Logic.

QUESTION 6 (EXTRA 15pts) Let **GL** be the sound Gentzen style proof system for classical logic defined in chapter 11.

1. Define SHORTLY Decomposition Tree for any A in **GL**.

2. Is the Tree Unique?

3. Use 1 and 2 to prove the Completeness Theorem for **GL**. We assume that the STRONG soundness has been proved.