CSE/MAT371 QUIZ 3 SOLUTIONS Fall 2016

Problem 1 (5pts) Complete the following definition.

Given a non-empty set $\mathcal{G} \subseteq \mathcal{F}$ of a language $\mathcal{L} = \mathcal{L}_{\{\neg,\Rightarrow\}}$, a truth assignment $v : VAR \longrightarrow \{T, F\}$, and classical semantics, we say that v is a **model** for the set $\mathcal{G}$ and write it $v \models \mathcal{G}$ if and only if ...

Solution We say: $v \models \mathcal{G}$ if and only if $v \models A$ (or $v^*(A) = T$) for all formulas $A \in \mathcal{G}$.

Problem 2 (8pts) Given a language $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$. We know that $v : VAR \longrightarrow \{F, \perp, T\}$ is such that $v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp$ under **H** semantics defined as follows.

For any $(x, y) \in \{T, \perp, F\} \times \{T, \perp, F\}$ we put $\cup(x, y) = x \cup y = max\{x, y\}$, $\cap(x, y) = x \cap y = min\{x, y\}$,

$\Rightarrow (x, y) = x \Rightarrow y = \begin{cases} T & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$ and for any $x \in \{T, \perp, F\}$ we put $\neg x = x \Rightarrow F$

Evaluate $v^*(A)$ for a formula A: $((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b)$

Use shorthand notation.

Solution We write TTables

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	F	T	T
T	F	$\perp$	T

$\neg$	F	$\perp$	T
	T	*F*	F

and observe that $v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp$ under **H** semantics if and only if (using a shorthand notation) $(a \cap b) = T$ and $(a \Rightarrow c) = \perp$ if and only if $a = T$, $b = T$ and $(T \Rightarrow c) = \perp$ if and only if $c = \perp$

I.e. we have that $v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp$ if and only if $a = T$, $b = T$, $c = \perp$

Now we can we **evaluate**

$v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b)) = (((T \Rightarrow T) \Rightarrow (T \Rightarrow \neg \perp)) \cup (T \Rightarrow T)) = ((T \Rightarrow (T \Rightarrow F)) \cup T) = T$

Problem 3

1. Given a formula $A = ((a \cap \neg c) \Rightarrow (\neg a \cup b))$ of a language $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$.

Find a formula B of a language $\mathcal{L}_{\{\neg,\Rightarrow\}}$, such that $A \equiv B$. **List** all proper logical equivalences used at at each step.

Solution :

$$A = ((a \cap \neg c) \Rightarrow (\neg a \cup b)) \equiv ((a \cap \neg c) \Rightarrow (a \Rightarrow b)) \equiv (\neg(a \Rightarrow \neg\neg c) \Rightarrow (a \Rightarrow b)) \equiv (\neg(a \Rightarrow c) \Rightarrow (a \Rightarrow b)) = B$$

Equivalences used: **1.** $(\neg A \cup B) \equiv (A \Rightarrow B)$, **2.** $(A \cap B) \equiv \neg(A \Rightarrow \neg B)$, **3.** $\neg\neg A \equiv A$.

2. Prove that $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}} \equiv \mathcal{L}_{\{\neg,\Rightarrow\}}$

Solution We have to prove that $\mathcal{L}_{\{\neg,\Rightarrow\}} \equiv \mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$.

Condition **C1** holds because $\{\neg, \Rightarrow\} \subseteq \{\neg, \cap, \cup, \Rightarrow\}$.

Condition **C2** holds because of the **Substitution Theorem** and because of the following **logical equivalences** for any for any formulas A, B

$$(A \cap B) \equiv \neg(A \Rightarrow \neg B) \quad \text{and} \quad (A \cup B) \equiv (\neg A \Rightarrow B)$$

Reminder We define the **equivalence of languages** as follows:

Given two languages: $\mathcal{L}_1 = \mathcal{L}_{CON_1}$ and $\mathcal{L}_2 = \mathcal{L}_{CON_2}$, for $CON_1 \neq CON_2$, we say that they are **logically equivalent**, i.e. $\mathcal{L}_1 \equiv \mathcal{L}_2$ if and only if the following conditions **C1, C2** hold.

C1: For every formula A of $\mathcal{L}_1$, there is a formula B of $\mathcal{L}_2$, such that $A \equiv B$,

C2: For every formula C of $\mathcal{L}_2$, there is a formula D of $\mathcal{L}_1$, such that $C \equiv D$.