CSE371 MIDTERM 1 SOLUTIONS Fall 2017

PART 1: DEFINITIONS

D1 Given a language $\mathcal{L}_{\{\Rightarrow,\cup,\cap,\neg\}}$ and a formula A of this language.

$\models A$ if and only if $v \models A$ for all **truth assignment** $v : VAR \longrightarrow \{T, F\}$

D2 Given formula $A \in \mathcal{F}$ of $\mathcal{L}_{\{\Rightarrow,\cup,\cap,\neg\}}$.

Write definition of v is a restricted model for A.

A **restricted MODEL** for the formula A is any function $w : VAR_A \longrightarrow \{T, F\}$ such that $w^*(A) = T$, where

VAR_A is the sent of all propositional variables appearing in A.

D3 Given a proof system $S = (\mathcal{L},\ \mathcal{E},\ LA,\ \mathcal{R})$ and an expression $E \in \mathcal{E}$.

$\vdash_S E$ if and only if there is a sequence $E_1,\ E_2, ,\ E_n$ of expressions from $\mathcal{E}$, such that $n \geq 1$, and for each $1 < i \leq n$, either $E_i \in LA$ or E_i is a **direct consequence** of some of the preceding expressions in $E_1,\ E_2, ,\ E_n$ by virtue of one of the rules of inference $r \in \mathcal{R}$.

D4 A proof system $S = (\mathcal{L},\ \mathcal{E},\ LA,\ \mathcal{R})$ is **complete** under a semantics **M** if and only if the following holds for any expression $E \in \mathcal{E}$.

$$\vdash_S E \quad \text{if and only if} \quad \models_M E.$$

D5 Write definition: A non-empty set $\mathcal{G} \subseteq \mathcal{F}$ **consistent** under classical semantics.

A non-empty set $\mathcal{G} \subseteq \mathcal{F}$ of **formulas** is called **consistent** if and only if $\mathcal{G}$ **has a model**.

We can also say:

$\mathcal{G} \subseteq \mathcal{F}$ is **consistent** if and only if there is a truth assignment v such that $v \models \mathcal{G}$,

or we say:

$\mathcal{G} \subseteq \mathcal{F}$ is **consistent** if and only if there is v is such that $v^*(A) = T$ for all $A \in \mathcal{G}$

PART 2: PROBLEMS

PROBLEM 1

Write the following natural language statement:

One likes to play bridge, or from the fact that the weather is good we conclude the following: one does not like to play bridge or one likes not to play bridge

as a formula of 2 different languages

1. Formula $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$, where **L** A represents statement "one likes A", "A is liked".

Solution We translate our statement into a formula

$A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$ as follows.

Propositional Variables: a, b
a denotes statement: *play bridge*,
b denotes a statement: *the weather is good*

Translation 1

$$A_1 = (\mathbf{L}a \cup (b \Rightarrow (\neg \mathbf{I}a \cup \mathbf{L}\neg a)))$$

2. Formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg,\ \cup,\ \Rightarrow\}}$.

Solution We translate our statement into a formula $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg,\ \cup,\ \Rightarrow\}}$ as follows.

Propositional Variables: a, b, c
a denotes statement: *One likes to play bridge* ,
b denotes a statement: *the weather is good*, and
c denotes a statement: *one likes not to play bridge*

Translation 2:

$$A_2 = (a \cup (b \Rightarrow (\neg a \cup c)))$$

PROBLEM 2

Given a formula A : $\forall x \exists y\ P(f(x, y), c)$ of the predicate language $\mathcal{L}$, and

two **model structures** $\mathbf{M_1} = (Z, I_1)$, $\mathbf{M_1} = (N, I_2)$ with the interpretations defined as follows.

$P_{I_1} := ,\quad f_{I_1} : \ + ,\quad c_{I_1} : \ 0$ and $P_{I_2} : > ,\quad f_{I_2} : \ \cdot ,\quad c_{I_2} : \ 0.$

1. Show that $\mathbf{M_1} \models A$

Solution

A_{I_1} : $\forall_{x \in Z} \exists_{y \in Z}\ x + y = 0$ is a **true** statement;

For each $x \in Z$ exists $y = -x$ and $-x \in Z$ and $x - x = 0$.

Solution

2. Show that $\mathbf{M_2} \not\models A$.

Solution

A_{I_2} : $\forall_{x \in N} \exists_{y \in N}\ x \cdot y > 0$ is a **false** statement for $x = 0$.

PROBLEM 3

We define a 3 valued extensional semantics $\mathbf{M}$ for the language $\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}}$ by **defining the connectives** $\neg$, $\cup$, $\Rightarrow$ on a set $\{F, \bot, T\}$ of logical values by the following truth tables.

L Connective

L	F	$\bot$	T
	F	F	T

Negation :

$\neg$	F	$\bot$	T
	T	F	F

Implication **Disjunction** :

$\Rightarrow$	F	$\bot$	T
F	T	T	T
$\bot$	T	$\bot$	T
T	F	F	T

$\cup$	F	$\bot$	T
F	F	$\bot$	T
$\bot$	$\bot$	T	T
T	T	T	T

1. Verify whether $\models_{\mathbf{M}} (\mathbf{L}A \cup \neg\mathbf{L}A)$. You can use **shorthand notation**.

Solution

We verify

$\mathbf{L}T \cup \neg\mathbf{L}T = T \cup F = T, \quad \mathbf{L}\bot \cup \neg\mathbf{L}\bot = F \cup \neg F = F \cup T = T, \quad \mathbf{L}F \cup \neg\mathbf{L}F = F \cup \neg F = T$

2. Verify whether your formulas A_1 and A_2 from PROBLEM 1 have a model/ counter model under the semantics **M**. You can use **shorthand notation**.

Solution

The formulas are: $A_1 = (\mathbf{L}a \cup (b \Rightarrow (\neg\mathbf{I}a \cup \mathbf{L}\neg a)))$, and $A_2 = (a \cup (b \Rightarrow (\neg a \cup c)))$.

Any v, such that $v(a) = T$ is a **M model** for A_1 and for A_2 directly from the definition of $\cup$.

3. Verify whether the following set **G** is **M**-consistent. You can use **shorthand notation**

$$\mathbf{G} = \{ \ \mathbf{L}a, \ \ (a \cup \neg\mathbf{L}b), \ \ (a \Rightarrow b), \ b \ \}$$

Solution

Any v, such that $v(a) = T, \ v(b) = T$ is a **M model** for **G** as

$$\mathbf{L}T = T, \quad (T \cup \neg\mathbf{L}T) = T, \quad (T \Rightarrow T) = T, \quad b = T.$$

PROBLEM 4

Let S be the following **proof system**

$$S = (\ \mathcal{L}_{\{\neg,\mathbf{L},\cup,\Rightarrow\}}, \ \mathcal{F}, \ \ \{\mathbf{A1}, \mathbf{A2}\}, \ \ \{(r1), \ (r2)\} \ )$$

for the logical axioms and rules of inference defined for any formulas $A, B \in \mathcal{F}$ as follows

Logical Axioms

A1 $(\mathbf{L}A \cup \neg\mathbf{L}A)$

A2 $(A \Rightarrow \mathbf{L}A)$

Rules of inference:

$$(r1) \ \frac{A \ ; \ B}{(A \cup B)}, \qquad (r2) \ \frac{A}{\mathbf{L}(A \Rightarrow B)}$$

1. Verify whether the system S is **M**-sound. You can use **shorthand notation**

Solution

Observe that both logical axioms of S are **M tautologies**

A1, A2 are **M** tautologies by direct evaluation.

Rule (r1) is **sound** because when $A = T$ and $B = T$ we get $A \cup B = T \cup T = T$.

Rule (r2) is **not sound** because when $A = T$ and $B = F$ (or $B = \perp$) we get $\mathbf{L}(A \Rightarrow B) = \mathbf{L}(T \Rightarrow F) = \mathbf{L}F = F$ or $\mathbf{L}(T \Rightarrow \perp) = \mathbf{L} \perp = F$

We proved that S is **not sound**.

2. Show, by constructing a proper **formal proof** that

$$\vdash_S ((\mathbf{L}b \cup \neg \mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg \mathbf{L}a) \Rightarrow b)))$$

You must write comments how each step pot the proof was obtained

Write all steps of the **formal proof** as follows - write as MANY as you NEED!

Solution

Here is the proof B_1, B_2, B_3, B_4 of $((\mathbf{L}b \cup \neg \mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg \mathbf{L}a) \Rightarrow b)))$.

B_1: $(\mathbf{L}a \cup \neg \mathbf{L}a)$ Axiom A_1 for A= a

B_2: $\mathbf{L}((\mathbf{L}a \cup \neg \mathbf{L}a) \Rightarrow b)$ rule (r2) for B=b applied to B_1

B_3: $(\mathbf{L}b \cup \neg \mathbf{L}Ab)$ Axiom A_1 for A=b

B_4: $((\mathbf{L}b \cup \neg \mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg \mathbf{L}a) \Rightarrow b))$ (r1) applied to B_3 and B_2.

3. Does above point **2.** prove that $\models ((\mathbf{L}b \cup \neg \mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg \mathbf{L}a) \Rightarrow b)))$?

Solution

No, , it doesn't because the system S is **not sound**

PROBLEM 5

Consider the Hilbert system $H_1 == (\mathcal{L}_{\{\Rightarrow\}}, \mathcal{F}, \{A1, A2\}, (MP) \frac{A \ ; \ (A \Rightarrow B)}{B})$ where

$A1;$ $(A \Rightarrow (B \Rightarrow A))$, $A2:$ $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$ and A, B are any formulas from $\mathcal{F}$.

Use **Deduction Theorem** to prove $\vdash_{H_1} ((B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$.

Write comments how each step was obtained.

Solution

By Deduction Theorem applied THREE times we get that

$\vdash_{H_1} ((B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$ if and only if

$(B \Rightarrow C) \vdash_{H_1} ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ if and only if

$(B \Rightarrow C), (A \Rightarrow B) \vdash_{H_1} (A \Rightarrow C)$ if and only if

$(B \Rightarrow C),\ (A \Rightarrow B),\ A \vdash_{H_1} C)$.

Here is a formal proof of C from $(A \Rightarrow B),\ (B \Rightarrow C),\ A$:

B_1 $(A \Rightarrow B)$ Hyp

B_2 A Hyp

B_3 B MP on B_2, B_1

B_4 $(B \Rightarrow C)$ Hyp

B_5 C MP on B_3, B_4