

cse371/math371  
LOGIC

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## LECTURE 4a

## Chapter 4 Review

**PART 1:** DEFINITIONS

**PART 2:** Problems

## PART 1: Definitions from Chapter 4 you have to know

## Definition: Proof System

### Definition 1

By a **proof system** we understand a quadruple

$$S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$$

where

$\mathcal{L} = \{\mathcal{A}, \mathcal{F}\}$  is a **language** of S with a set  $\mathcal{F}$  of formulas

$\mathcal{E}$  is a set of **expressions** of S

In particular case  $\mathcal{E} = \mathcal{F}$

$LA \subseteq \mathcal{E}$  is a **non-empty, finite** set of **logical axioms** of S

$\mathcal{R}$  is a **non-empty, finite set** of **rules of inference** of S

## Definition: Sound Rule of Inference

### Definition 2

An inference rule

$$(r) \quad \frac{P_1 ; P_2 ; \dots ; P_m}{C}$$

is **sound** under a semantics **M** if and only if all **M** - models of the set  $\{P_1, P_2, \dots, P_m\}$  of its **premisses** are also **M** - models of its **conclusion C**

In particular, in case of **extensional propositional semantics** when the condition below holds for any truth assignment  $v : VAR \rightarrow LV$

**If**  $v \models_M \{P_1, P_2, \dots, P_m\}$ , **then**  $v \models_M C$

## Definition: Direct Consequence

### Definition 3

For any rule of inference  $r \in \mathcal{R}$  of the form

$$(r) \quad \frac{P_1 ; P_2 ; \dots ; P_m}{C}$$

$C$  is called a **direct consequence** of  $P_1, \dots, P_m$  by virtue of the rule  $r \in \mathcal{R}$

## Definition: Formal Proof

### Definition 4

A **formal proof** of an expression  $E \in \mathcal{E}$  in a proof system  $S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$  is a sequence

$$A_1, A_2, \dots, A_n \text{ for } n \geq 1$$

of expressions from  $\mathcal{E}$ , such that

$$A_1 \in LA, \quad A_n = E$$

and for each  $1 < i \leq n$ , either  $A_i \in LA$  or  $A_i$  is a **direct consequence** of some of the **preceding expressions** by virtue of **one** of the **rules of inference**

$n \geq 1$  is the **length** of the proof  $A_1, A_2, \dots, A_n$



## NOTATION: Provable Expressions

### Notation

We write  $\vdash_S E$  to denote that  $E \in \mathcal{E}$  **has a formal proof** in the proof system  $S$

A set

$$\mathbf{P}_S = \{E \in \mathcal{E} : \vdash_S E\}$$

is called the set of **all provable expressions** in  $S$

## Definition: Sound $S$

### Definition 5

Given a proof system

$$S = (\mathcal{L}, \mathcal{E}, LA, \mathcal{R})$$

We say that the system  $S$  is **sound** under a semantics  $\mathbf{M}$  iff the following conditions hold

1. Logical axioms  $LA$  are **tautologies** of under the semantics  $\mathbf{M}$ , i.e.

$$LA \subseteq \mathbf{T}_M$$

2. Each **rule of inference**  $r \in \mathcal{R}$  is **sound** under the semantics  $\mathbf{M}$

## THEOREMS: Soundness Theorem

Let  $\mathbf{P}_S$  be the set of all provable expressions of  $\mathbf{S}$  i.e.

$$\mathbf{P}_S = \{A \in \mathcal{E} : \vdash_S A\}$$

Let  $\mathbf{T}_M$  be a set of all expressions of  $\mathbf{S}$  that are **tautologies** under a semantics  $\mathbf{M}$ , i.e.

$$\mathbf{T}_M = \{A \in \mathcal{E} : \models_M A\}$$

Our GOAL is to prove the following theorems:

**Soundness Theorem** ( for  $\mathbf{S}$  and semantics  $\mathbf{M}$  )

$$\mathbf{P}_S \subseteq \mathbf{T}_M$$

i.e. for any  $A \in \mathcal{E}$ , the following implication holds

$$\text{If } \vdash_S A \text{ then } \models_M A$$

## THEOREMS: Completeness Theorem

**Completeness Theorem** (for **S** and semantics **M**)

$$P_S = T_M$$

i.e. for any  $A \in \mathcal{E}$ , the following holds

$$\vdash_S A \quad \text{if and only if} \quad \models_M A$$

The **Completeness Theorem** consists of two parts:

**Part 1: Soundness Theorem**

$$P_S \subseteq T_M$$

**Part 2: Completeness Part** of the Completeness Theorem

$$T_M \subseteq P_S$$

## PART 2: Simple Problems

## Formal Proofs

### Problem 1

Given a proof system:

$$S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, \mathcal{R} = \{(r)\})$$

$$\text{where } (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}$$

Write a **formal proof** in  $S$  with 2 applications of the rule  $(r)$

**Solution:** There are many solutions. Here is one of them.

Required formal proof is a sequence  $A_1, A_2, A_3$ , where

$$A_1 = (A \Rightarrow A)$$

(Axiom)

$$A_2 = (A \Rightarrow (A \Rightarrow A))$$

Rule  $(r)$  application 1 for  $A = A, B = A$

$$A_3 = ((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A)))$$

Rule  $(r)$  application 2 for  $A = A, B = (A \Rightarrow A)$

## Soudness

Given a proof system:

$$S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))})$$

### Problem 2

Prove that **S** is **sound** under classical semantics.

### Solution

1. Both axioms of **S** are basic classical tautologies
2. Consider the rule of inference of **S**

$$(r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}$$

Assume that its premise (the only premise) is true, i.e. let  $v$  be any truth assignment, such that  $v^*(A \Rightarrow B) = T$

We evaluate logical value of the conclusion under the truth assignment  $v$  as follows

$$v^*(B \Rightarrow (A \Rightarrow B)) = v^*(B) \Rightarrow T = T$$

for any  $B$  and any value of  $v^*(B)$

## Formal Proof

Given a proof system:

$$S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))})$$

### Problem 3.

Write a **formal proof** of your choice in  $S$  with 2 applications of the rule  $(r)$

### Solution

There many of such proofs, of different length, with different choice if axioms - here is my choice:  $A_1, A_2, A_3$ , where

$$A_1 = (A \Rightarrow A)$$

(Axiom)

$$A_2 = (A \Rightarrow (A \Rightarrow A))$$

Rule  $(r)$  application 1 for  $A = A, B = A$

$$A_3 = ((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A)))$$

Rule  $(r)$  application 2 for  $A = A, B = (A \Rightarrow A)$



## Formal Proof

Given a proof system:

$$S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))})$$

### Problem 4

1. Prove, by constructing a **formal proof** that

$$\vdash_S ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))$$

**Solution** Required formal proof is a sequence  $A_1, A_2$ ,  
where

$$A_1 = (A \Rightarrow (\neg A \Rightarrow B))$$

Axiom

$$A_2 = ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))$$

Rule  $(r)$  application for  $A = A, B = (\neg A \Rightarrow B)$

## Soundness Theorem

2. Does above point 1. prove that

$$\models ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))?$$

### Solution

**Yes**, it does because the system **S** is **sound** and we proved by Mathematical Induction over the length of a proof that if **S** is **sound**, then the **Soundness Theorem** holds for **S**

## Soundness

### Problem 5

Given a proof system:

$$S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \mathcal{F}, \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))})$$

Prove that **S** is **not sound** under **K** semantics

### Solution

Axiom  $(A \Rightarrow A)$  is not a **K** semantics tautology

Any truth assignment  $v$  such that  $v^*(A) = \perp$  is a **counter-model** for it

This proves that **S** is **not sound** under **K** semantics