cse371/Math371 LOGIC

Professor Anita Wasilewska

LECTURE 6c

Chapter 5, Chapter 6
REVIEW for Q2

Chapter 5: System H_2 and examples of **formal proofs** in H_2

Chapter 6: Proof Systems RS, RS1, RS2

Chapter 6: Proof Systems GL, G

CHAPTER 5
Hilbert System H_2 Definition

Definition

$$H_2 = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\ \mathcal{F},\ \{A1, A2, A3\}\ \ (MP)\ )$$

A1 (Law of simplification)
$(A \Rightarrow (B \Rightarrow A))$

A2 (Frege's Law)
$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

where A, B, C are any formulas of the propositional language $\mathcal{L}_{\{\Rightarrow,\neg\}}$

Deduction Theorem for H_2

Deduction Theorem for H_2

For any $A, B \in \mathcal{F}$ and $\Gamma \subseteq \mathcal{F}$

$$\Gamma,\ A \vdash_{H_1} B \quad \text{if and only if} \quad \Gamma \vdash_{H_2} (A \Rightarrow B)$$

In particular

$$A \vdash_{H_2} B \quad \text{if and only if} \quad \vdash_{H_2} (A \Rightarrow B)$$

Formal Proofs

The proof of the following **Lemma** provides a good example of multiple applications of the **Deduction Theorem**

Lemma

For any $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$,

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_{H_2} (B \Rightarrow (A \Rightarrow C))$

Observe that by **Deduction Theorem** we can re-write **(a)** as

(a') $(A \Rightarrow B),\ (B \Rightarrow C), A \vdash_{H_2} C$

Soundness and CompletenessTheorems

We get by easy verification

Soundness Theorem H_2

For every formula $A \in \mathcal{F}$

$$\text{if } \vdash_{H_2} A \quad \text{then} \quad \models A$$

We prove in the next Lecture, that H_2 is also complete, i.e. we prove

Completeness Theorem for H_2

For every formula $A \in \mathcal{F}$,

$$\vdash_{H_2} A \ \text{ if and only if } \ \models A$$

FORMAL PROOFS IN H_2

Examples and Exercises

We present now some examples of **formal proofs** in H_2

There are two reasons for presenting them.

First reason is that all formulas we prove here to be provable play a crucial role in the **proof** of Completeness Theorem for H_2

The second reason is that they provide a ”training ground” for a reader to learn how to develop formal proofs

For this reason we write some proofs in a full detail and we leave some for the reader to complete in a way explained in the following example.

Important Lemma

We write $\vdash$ instead of $\vdash_{H_2}$ for the sake of simplicity

Reminder

In the construction of the formal proofs we often use the **Deduction Theorem** and the following **Lemma 1** they was proved in previous section

Lemma 1

$$(\mathbf{a})\ \ (A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$$

$$(\mathbf{b})\ \ (A \Rightarrow (B \Rightarrow C)) \vdash_{H_2} ((B \Rightarrow (A \Rightarrow C))$$

Example 1

Example 1

Here are consecutive steps

$$B_1, ..., B_5, \ B_6$$

of the proof in H_2 of $(\neg\neg B \Rightarrow B)$

B_1 : $((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$

B_2 : $((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

B_3 : $(\neg B \Rightarrow \neg B)$

B_4 : $((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$

B_5 : $(\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$

B_6 : $(\neg\neg B \Rightarrow B)$

Exercise 1

Exercise 1

Complete the proof presented in Example 1 by providing comments how each step of the proof was obtained.

ATTENTION

The solution presented on the next slide shows you how you will have to write details of your solutions on the **TESTS**

Solutions of other problems presented later are less detailed

Use them as exercises to write a detailed, complete solutions

Exercise 1 Solution

Solution

The comments that complete the proof are as follows.

B_1 : $((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$
Axiom $A3$ for $A = \neg B, B = B$

B_2 : $((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$
B_1 and **Lemma 1 (b)** for
$A = (\neg B \Rightarrow \neg\neg B), \ B = (\neg B \Rightarrow \neg B), C = B$, i.e. we have
$((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B)) \vdash ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

Exercise 1 Solution

$B_3 : \quad (\neg B \Rightarrow \neg B)$

We proved for H_1 and hence for H_2 that $\vdash (A \Rightarrow A)$ and we substitute $A = \neg B$

$B_4 : \quad ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$

B_2, B_3 and MP

$B_5 : \quad (\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$

Axiom $A1$ for $A = \neg\neg B, B = \neg B$

$B_6 : \quad (\neg\neg B \Rightarrow B)$

B_4, B_5 and **Lemma 1 (a)** for

$A = \neg\neg B, B = (\neg B \Rightarrow \neg\neg B), C = B$; i.e.

$(\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B)),\ ((\neg B \Rightarrow \neg\neg B) \Rightarrow B) \vdash (\neg\neg B \Rightarrow B)$

Proofs from Axioms Only

General remark

Observe that in steps B_2, B_3, B_5, B_6 we **call on** previously proved facts and use them as a part of our proof.

We can **obtain** a proof that uses only axioms by inserting previously constructed formal proofs of these facts into the places occupying by the steps B_2, B_3, B_5, B_6

For example in previously constructed proof of $(A \Rightarrow A)$ we **replace** A by $\neg B$ and **insert** such constructed proof of $(\neg B \Rightarrow \neg B)$ after step B_2

The last step of the inserted proof becomes now ”old” step B_3 and we re-numerate all other steps accordingly

Proofs from Axioms Only

Here are consecutive first THREE steps of the proof of $(\neg\neg B \Rightarrow B)$

B_1 : $((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$

B_2 : $((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

B_3 : $(\neg B \Rightarrow \neg B)$

We insert now the proof of $(\neg B \Rightarrow \neg B)$ after step B_2 and **erase** the B_3

The **last step** of the inserted proof becomes the **erased** B_3

Proofs from Axioms Only

A part of new **transformed** proof is

B_1 : $((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$ (Old B_1)

B_2 : $((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$ (Old B_2)

We insert here the proof from axioms only of Old B_3

B_3 : $((\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B)) \Rightarrow ((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B)))$, (New B_3)

B_4 : $(\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B))$

B_5: $((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B)))$

B_6: $(\neg B \Rightarrow (\neg B \Rightarrow \neg B))$

B_7: $(\neg B \Rightarrow \neg B)$ (Old B_3)

Proofs from Axioms Only

We repeat our procedure by **replacing** the step B_2 by its formal proof as defined in the proof of the **Lemma 1 (b)**

We continue the process for all other steps which involved application of the **Lemma 1** until we get a full **formal proof** from the axioms of H_2 only

Usually we don't do it and we don't need to do it, but it is important to remember that **it always can be done**

Example 2

Example 2

Here are consecutive steps

$$B_1, \ B_2, \ \ \ldots.., \ \ B_5$$

in a proof of $(B \Rightarrow \neg\neg B)$

B_1 $((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$

B_2 $(\neg\neg\neg B \Rightarrow \neg B)$

B_3 $((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$

B_4 $(B \Rightarrow (\neg\neg\neg B \Rightarrow B))$

B_5 $(B \Rightarrow \neg\neg B)$

Exercise 2

Exercise 2

Complete the proof presented in **Example 2** by providing detailed comments how each step of the proof was obtained.

Solution

The comments that complete the proof are as follows.

B_1 $((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$
Axiom $A3$ for $A = B, B = \neg\neg B$

B_2 $(\neg\neg\neg B \Rightarrow \neg B)$
Example 1 for $B = \neg B$

Exercise 2

B_3 $((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$

B_1, B_2 and MP, i.e.

$$\frac{(\neg\neg\neg B \Rightarrow \neg B);((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))}{((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)}$$

B_4 $(B \Rightarrow (\neg\neg\neg B \Rightarrow B))$

Axiom $A1$ for $A = B, \quad B = \neg\neg\neg B$

B_5 $(B \Rightarrow \neg\neg B)$

B_3, B_4 and lemma 1**a** for $A = B, B = (\neg\neg\neg B \Rightarrow B), C = \neg\neg B$, i.e.

$$(B \Rightarrow (\neg\neg\neg B \Rightarrow B)), ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B) \vdash (B \Rightarrow \neg\neg B)$$

CHAPTER 6
RS Proof Systems

RS Decomposition Rules and Decomposition Trees

Decomposition Trees

The process of searching for a proof of a formula $A \in \mathcal{F}$ in **RS** consists of building a certain tree $\mathbf{T}_A$, called a **decomposition tree**

Building a **decomposition tree** what really is a proof search **tree** consists in the **first step** of transforming the **RS rules** into corresponding decomposition **rules**

Tree Rules

We write the **Decomposition Rules** in a visual tree form as follows

Tree Rules

$(\cup)$ **rule**

$$\Gamma^{'},\ (A \cup B),\ \Delta$$

$$\mid (\cup)$$

$$\Gamma^{'},\ A, B,\ \Delta$$

Tree Rules

$(\neg\cup)$ rule

$$\Gamma', \neg(A \cup B), \Delta$$

$$\bigwedge(\neg\cup)$$

$$\Gamma', \neg A, \Delta \qquad\qquad \Gamma', \neg B, \Delta$$

$(\cap)$ rule

$$\Gamma', (A \cap B), \Delta$$

$$\bigwedge(\cap)$$

$$\Gamma', A, \Delta \qquad\qquad \Gamma', B, \Delta$$

Tree Rules

$(\neg\cup)$ **rule**

$$\Gamma^{'}, \ \neg(A \cap B), \ \Delta$$

$$\mid (\neg\cap)$$

$$\Gamma^{'}, \ \neg A, \neg B, \ \Delta$$

$(\Rightarrow)$ **rule**

$$\Gamma^{'}, \ (A \Rightarrow B), \ \Delta$$

$$\mid (\Rightarrow)$$

$$\Gamma^{'}, \ \neg A, B, \ \Delta$$

Tree Rules

$(\neg \Rightarrow)$ **rule**

$$\Gamma', \neg(A \Rightarrow B), \Delta$$

$$\bigwedge (\neg \Rightarrow)$$

$$\Gamma', A, \Delta \qquad\qquad \Gamma', \neg B, \Delta$$

$(\neg\neg)$ **rule**

$$\Gamma', \neg\neg A, \Delta$$

$$\mid (\neg\neg)$$

$$\Gamma', A, \Delta$$

Definitions and Observations

Observe that we use the same names for the **inference** and **decomposition** rules

We do so because once the we have built the **decomposition tree** with all leaves being **axioms**, it constitutes a **proof** of *A* in **RS** with branches labeled by the proper **inference rules**

Now we still need to introduce few standard and useful definitions and observations.

Definitions and Observations

Definition

A sequence Γ' built only out of literals, i.e. $\Gamma \in \mathcal{F}'^*$ is called an **indecomposable sequence**

Definition

A formula A that is not a literal, i.e. $A \in \mathcal{F} - LT$ is called a **decomposable formula**

Definition

A sequence Γ that contains a decomposable formula is called a

decomposable sequence

Definitions and Observations

Observation 1

For any **decomposable** sequence, i.e. for any $\Gamma \notin LT^*$ there is **exactly one** decomposition rule that can be applied to it

This rule is **determined** by the first **decomposable formula** in Γ and by the main connective of that formula

Definitions and Observations

Observation 2

If the main connective of the **first** decomposable formula is $\cup, \cap, \Rightarrow$, then the **decomposition rule** determined by it is $(\cup), (\cap), (\Rightarrow)$, respectively

Observation 3

If the main connective of the **first** decomposable formula A is negation $\neg$, then the **decomposition rule** is determined by the **second connective** of the formula A

The corresponding **decomposition rules** are $(\neg\cup), (\neg\cap), (\neg\neg), (\neg\Rightarrow)$

Decomposition Lemma

Because of the importance of the **Observation 1** we re-write it in a form of the following

Decomposition Lemma

For any sequence $\Gamma \in \mathcal{F}^*$,

$\Gamma \in LT^*$ or Γ is in the domain of **exactly one** of **RS** Decomposition Rules

This rule is **determined** by the first decomposable formula in Γ and by the main connective of that formula

Decomposition Tree Definition

Definition: **Decomposition Tree** $\mathbf{T}_A$

For each $A \in \mathcal{F}$, a **decomposition tree** $\mathbf{T}_A$ is a tree build as follows

Step 1.

The formula A is the **root** of $\mathbf{T}_A$

For any other **node** Γ of the tree we follow the steps below

Step 2.

If Γ is **indecomposable** then Γ becomes a **leaf** of the tree

Decomposition Tree Definition

Step 3.

If Γ is **decomposable**, then we **traverse** Γ from **left** to **right** and identify the **first** **decomposable formula** B

By the **Decomposition Lemma**, there is exactly one decomposition rule determined by the main connective of B

We put its premiss as a **node below**, or its left and right premisses as the left and right **nodes below**, respectively

Step 4.

We repeat **Step 2** and **Step 3** until we obtain only leaves

Decomposition Theorem

We now prove the following **Decomposition Tree Theorem**. This Theorem provides a crucial step in the proof of the Completeness Theorem for RS

Decomposition Tree Theorem

For any sequence $\Gamma \in \mathcal{F}^*$ the following conditions hold

1. $\mathbf{T}_\Gamma$ is finite and unique

2. $\mathbf{T}_\Gamma$ is a proof of Γ in **RS** if and only if all its leafs are axioms

3. $\nvdash_{\mathbf{RS}} \Gamma$ if and only if $\mathbf{T}_\Gamma$ has a non- axiom leaf

Completeness Theorem

Our main goal is to prove the **Completeness Theorem** for **RS**

We **prove** first the following **Completeness Theorem** for formulas $A \in \mathcal{F}$

Completeness Theorem 1 For any formula $A \in \mathcal{F}$

$$\vdash_{\mathbf{RS}} A \quad \text{if and only if} \quad \models A$$

and then we generalize it to the following

Completeness Theorem 2 For any $\Gamma \in \mathcal{F}^*$,

$$\vdash_{\mathbf{RS}} \Gamma \quad \text{if and only if} \quad \models \Gamma$$

Do do so we need to introduce a new notion of a Strong Soundness and prove that the **RS** is strongly sound

Part 2: Strong Soundness and Constructive Completeness

Strong Soundness

Definition

Given a proof system

$$S = (\mathcal{L}, \ \mathcal{E}, \ LA, \ \mathcal{R})$$

Definition

A rule $r \in \mathcal{R}$ such that the **conjunction** of all its premisses is **logically equivalent** to its conclusion is called **strongly sound**

Definition

A proof system S is called **strongly sound** if and only if S is sound and **all** its rules $r \in \mathcal{R}$ are **strongly sound**

Strong Soundness of RS

Theorem

The proof system **RS** is strongly sound

Proof

We prove as an example the **strong soundness** of two of inference rules: $(\cup)$ and $(\neg\cup)$

Proof for all other rules follows the same patterns and is left as an exercise

By definition of strong soundness we have to show that

If P_1, P_2 are premisses of a given rule and C is its conclusion, then for all v,

$$v^*(P_1) = v^*(C)$$

in case of one premiss rule and

$$v^*(P_1) \cap v^*(P_2) = v^*(C)$$

in case of the two premisses rule.

Strong Soundness of RS

Consider the rule $(\cup)$

$$(\cup) \quad \frac{\Gamma^{'},\ A, B,\ \Delta}{\Gamma',\ (A \cup B),\ \Delta}$$

We evaluate:

$$v^{*}(\Gamma^{'}, A, B, \Delta) = v^{*}(\delta_{\{\Gamma', A, B, \Delta\}}) = v^{*}(\Gamma^{'}) \cup v^{*}(A) \cup v^{*}(B) \cup v^{*}(\Delta)$$

$$= v^{*}(\Gamma^{'}) \cup v^{*}(A \cup B) \cup v^{*}(\Delta) = v^{*}(\delta_{\{\Gamma', (A \cup B), \Delta\}})$$

$$= v^{*}(\Gamma^{'}, (A \cup B), \Delta)$$

Strong Soundness

We proved that all the rules of inference of **RS** of are **strongly sound**, i.e. $C \equiv P$ and $C \equiv P_1 \cap P_2$

Strong soundness of the rules hence means that if **at least** one of premisses of a rule is **false**, so is its conclusion

Given a formula A, such that its $\mathbf{T}_A$ has a branch ending with a non-axiom leaf

By **strong soundness**, any v that make this non-axiom leaf **false** also **falsifies** all sequences on that branch, and hence **falsifies the** the formula A

This means that any v that **falsifies** a non-axiom leaf is a **counter-model** for A

Counter Model Theorem

We have proved the following

Counter Model Theorem

Let $A \in \mathcal{F}$ be such that its decomposition tree $\mathbf{T}_A$ contains a **non- axiom** leaf L_A

Any truth assignment v that **falsifies** L_A is a **counter model** for A

Any truth assignment that **falsifies** a non- axiom **leaf** is called a **counter-model** for A **determined** by the decomposition tree $\mathbf{T}_A$

Counter Model Example

Consider a tree $\mathbf{T}_A$

$$(((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$$

$$\mid (\cup)$$

$$((a \Rightarrow b) \cap \neg c), (a \Rightarrow c)$$

$$\bigwedge(\cap)$$

$$(a \Rightarrow b), (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg a, b, (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg a, b, \neg a, c$$

$$\neg c, (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg c, \neg a, c$$

Counter Model Example

The tree $\mathbf{T}_A$ has a **non-axiom leaf**

$$L_A : \neg a,\ b, \neg a,\ c$$

We want to define a truth assignment $v : \ VAR \longrightarrow \{T, F\}$ **falsifies** this leaf L_A

Observe that v must be such that

$v^*(\neg a, b, \neg a, c) = v^*(\neg a) \cup v^*(b) \cup v^*(\neg a) \cup v^*(c) =$

$\neg v(a) \cup v(b) \cup \neg v(a) \cup v(c) = F$

It means that all components of the **disjunction** must be put to F

Counter Model Example

We hence get that v must be such that

$$v(a) = T, \ \ v(b) = F, \ \ \ v(c) = F$$

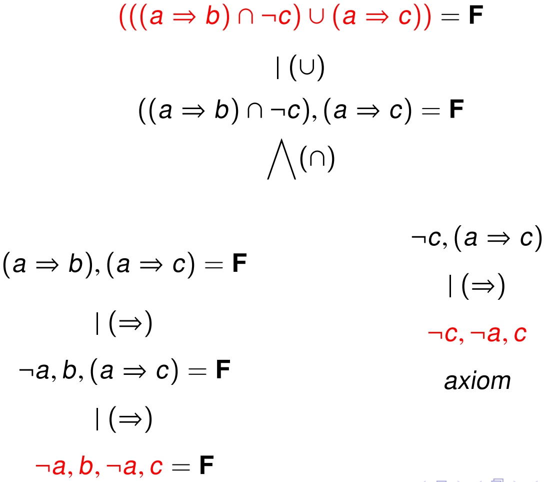
By the **Counter Model Theorem**, the v **determined** by the non-axiom leaf also **falsifies** the formula A

IT proves that v is a **counter model** for A and

$$\not\models (((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c))$$

Counter Model

The **Counter Model Theorem** says that **F** determined by the non-axiom leaf "climbs" the tree $\mathbf{T}_A$



$$(((a \Rightarrow b) \cap \neg c) \cup (a \Rightarrow c)) = \mathbf{F}$$

$$\mid (\cup)$$

$$((a \Rightarrow b) \cap \neg c), (a \Rightarrow c) = \mathbf{F}$$

$$\bigwedge (\cap)$$

$$(a \Rightarrow b), (a \Rightarrow c) = \mathbf{F}$$

$$\mid (\Rightarrow)$$

$$\neg a, b, (a \Rightarrow c) = \mathbf{F}$$

$$\mid (\Rightarrow)$$

$$\neg a, b, \neg a, c = \mathbf{F}$$

$$\neg c, (a \Rightarrow c)$$

$$\mid (\Rightarrow)$$

$$\neg c, \neg a, c$$

axiom

Counter Model

Observe that the same counter model **construction** applies to any other non-axiom **leaf**, if exists

The other non-axiom leaf defines another **F** that also ”climbs the tree” picture, and hence defines another **counter- model** for A

By **Decomposition Tree Theorem** all possible **restricted** counter-models for A are those **determined** by all non-axioms **leaves** of the $\mathbf{T}_A$

In our case the formula $\mathbf{T}_A$ has only one non-axiom leaf, and hence only one restricted **counter model**

RS Completeness Theorem

RS Completeness Theorem

For any $A \in \mathcal{F}$,

$$\text{If} \quad \models A, \quad \text{then} \quad \vdash_{\mathbf{RS}} A$$

We prove instead the opposite implication

RS Completeness Theorem

$$\text{If} \quad \nvdash_{\mathbf{RS}} A \quad \text{then} \quad \not\models A$$

Proof of Completeness Theorem

Proof of **Completeness Theorem**

Assume that A is any formula is such that

$$\nvdash_{\mathbf{RS}} A$$

By the **Decomposition Tree Theorem** the $\mathbf{T}_A$ contains a non-axiom leaf

The non-axiom leaf L_A **defines** a truth assignment v which **falsifies** it as follows:

$$v(a) = \begin{cases} F & \text{if } a \text{ appears in } L_A \\ T & \text{if } \neg a \text{ appears in } L_A \\ \text{any value} & \text{if } a \text{ does not appear in } L_A \end{cases}$$

Hence by **Counter Model Theorem** we have that v also **falsifies** A , i.e.

$$\not\models A$$

System **RS2** Definition

RS2 Definition

System **RS2** is a proof system obtained from **RS** by changing the sequences Γ' into Γ in **all of the rules** of inference of **RS**

The **logical axioms** LA remind the same

Observe that now the decomposition tree may not be unique

Exercise 1

Construct **two** decomposition trees in **RS2** of the formula

$$(\neg(\neg a \Rightarrow (a \cap \neg b)) \Rightarrow (\neg a \cap (\neg a \cup \neg b)))$$

RS2 Exercises

$\mathbf{T1}_A$

$(\neg(\neg a => (a \cap \neg b)) => (\neg a \cap (\neg a \cup \neg b)))$

$\mid (\Rightarrow)$

$\neg\neg(\neg a => (a \cap \neg b)), (\neg a \cap (\neg a \cup \neg b))$

$\mid (\neg\neg)$

$(\neg a => (a \cap \neg b)), (\neg a \cap (\neg a \cup \neg b))$

$\mid (\Rightarrow)$

$\neg\neg a, (a \cap \neg b), (\neg a \cap (\neg a \cup \neg b))$

$\mid (\neg\neg)$

$a, (a \cap \neg b), (\neg a \cap (\neg a \cup \neg b))$

$\bigwedge (\cap)$

$a, a, (\neg a \cap (\neg a \cup \neg b))$

$\bigwedge (\cap)$

$a, a.\neg a, (\neg a \cup \neg b)$

$\mid (\cup)$

$a, a.\neg a, \neg a, \neg b$

axiom

$a, a, (\neg a \cup \neg b)$

$\mid (\cup)$

$a, a, \neg a, \neg b$

axiom

$a, \neg b, (\neg a \cap (\neg a \cup \neg b))$

$\bigwedge (\cap)$

$a, \neg b, \neg a$

axiom

$a, \neg b, (\neg a \cup \neg b)$

$\mid (\cup)$

$a, \neg b, \neg a, \neg b$

axiom

RS2 Exercises

T2$_A$

$$(\neg(\neg a \Rightarrow (a \cap \neg b)) \Rightarrow (\neg a \cap (\neg a \cup \neg b)))$$

$$\mid (\Rightarrow)$$

$$\neg\neg(\neg a \Rightarrow (a \cap \neg b)), (\neg a \cap (\neg a \cup \neg b))$$

$$\mid (\neg\neg)$$

$$(\neg a \Rightarrow (a \cap \neg b)), (\neg a \cap (\neg a \cup \neg b))$$

$$\bigwedge (\cap)$$

$$(\neg a \Rightarrow (a \cap \neg b)), \neg a$$

$$\mid (\Rightarrow)$$

$$\neg\neg a, (a \cap \neg b), \neg a$$

$$\mid (\neg\neg)$$

$$a, (a \cap \neg b), \neg a$$

$$\bigwedge (\cap)$$

$$a, a, \neg a$$

axiom

$$a, \neg b, \neg a$$

axiom

$$(\neg a \Rightarrow (a \cap \neg b)), (\neg a \cup \neg b)$$

$$\mid (\cup)$$

$$(\neg a \Rightarrow (a \cap \neg b)), \neg a, \neg b$$

$$\mid (\Rightarrow)$$

$$\neg\neg a, (a \cap \neg b), \neg a, \neg b$$

$$\mid (\neg\neg)$$

$$a, (a \cap \neg b), \neg a, \neg b$$

$$\bigwedge (\cap)$$

$$a, a, \neg a, \neg b$$

axiom

$$a, \neg b, \neg a, \neg b$$

axiom

System **RS2**

Exercise 2

Explain why the system **RS2** is **strongly sound**. You can use the soundness of the system RS

Solution

The only difference between **RS** and **RS2** is that in **RS2** each inference rule has at the beginning a sequence of any formulas, not only of literals, as in **RS**

So there are many ways to **apply rules** as the decomposition rules while constructing the **decomposition tree**

But it does not affect **strong soundness**, since for all rules of **RS2** premisses and conclusions are still logically equivalent as they were in **RS**

RS2 Exercises

Exercise 3

Define shortly, in your own words, for any formula A, its **decomposition tree** $\mathbf{T}_A$ in **RS2**

Justify why your definition is correct

Show that in **RS2** the decomposition tree for some formula A may not be unique

RS2 Exercises

Solution

Given a formula A

The **decomposition tree** $\mathbf{T}_A$ can be defined as follows

It has the formula A as a **root**

For each **node**, if there is a **rule** of **RS2** which conclusion has the same form as **node** sequence, i.e.

if there is a **decomposition rule** to be applied, then the **node** has children that are premises of the **rule**

RS2 Exercises

If the **node** consists only of literals (i.e. there is no decomposition rule to be applied), then it **does not** have any children

The last statement defines a termination condition for the **tree**

This definition **correctly** defines a decomposition tree as it identifies and uses appropriate the **decomposition** rules

RS2 Exercises

Since in **RS2** **all** rules of inference have a sequence Γ instead of Γ' as it was defined for in **RS**, the **choice** of the decomposition rule for a node may be **not unique**

For **example** consider a **node**

$$(a => b), (b \cup a)$$

Γ in the **RS2** rules is a sequence of formulas, not literals, so for this **node** we can choose either rule $(=>)$ or rule $(\cup)$ as a **decomposition rule**

This leads to existence of non-unique **trees**

RS2 Exercises

Exercise 4

Prove the **Completeness Theorem** for **RS2**

Solution

We need to prove the completeness part only, as the soundness has been already proved, i.e. we have to prove the implication: for any formula A ,

$$\text{if} \quad \nvdash_{RS2} \; A \quad \text{then} \quad \not\models A$$

Assume $\nvdash_{RS2} A$,

Then **every** decomposition tree of A has at least one non-axiom **leaf**

Otherwise, there **would exist** a tree with all axiom leaves and it would be a **proof** for A

RS2 Exercises

Let $\mathcal{T}_A$ be a set of **all** decomposition trees of A

We choose an arbitrary $T_A \in \mathcal{T}_A$ with at least one non-axiom leaf L_A

The non-axiom leaf L_A **defines** a truth assignment v which falsifies A, as follows:

$$v(a) = \begin{cases} F & \text{if } a \text{ appears in } L_A \\ T & \text{if } \neg a \text{ appears in } L_A \\ \text{any value} & \text{if } a \text{ does not appear in } L_A \end{cases}$$

The value for a sequence that corresponds to the leaf in is F

Since, because of the **strong soundness** F ”climbs” the tree, we found a **counter-model** for A, i.e.

$$\not\models A$$

CHAPTER 6
Gentzen GL Proof Systems

Gentzen System **GL** Definition

Definition

$$\mathbf{GL} = (\ \mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}},\ \ SQ,\ \ LA,\ \ \mathcal{R}\ )$$

where

$$SQ = \{\ \Gamma \longrightarrow \Delta :\ \ \Gamma, \Delta \in \mathcal{F}^*\ \}$$

$$\mathcal{R} = \{(\cap \longrightarrow),\ (\longrightarrow \cap),\ (\cup \longrightarrow),\ (\longrightarrow \cup),\ (\Rightarrow \longrightarrow),\ (\longrightarrow \Rightarrow)\}$$

$$\cup \{(\neg \longrightarrow),\ (\longrightarrow \neg)\}$$

We write, as usual,

$$\vdash_{\mathbf{GL}} \Gamma \longrightarrow \Delta$$

to denote that $\Gamma \longrightarrow \Delta$ has a formal proof in **GL**

For any formula $A \in \mathcal{F}$

$$\vdash_{\mathbf{GL}} A \quad \text{if ad only if} \quad \longrightarrow A$$

Proof Trees

We consider, as we did with **RS** the proof trees for **GL**, i.e. we define

A **proof tree**, or **GL**-proof of $\Gamma \longrightarrow \Delta$ is a tree

$$\mathbf{T}_{\Gamma \longrightarrow \Delta}$$

of sequents satisfying the following conditions:

1. The topmost sequent, i.e **the root** of $\mathbf{T}_{\Gamma \longrightarrow \Delta}$ is $\Gamma \longrightarrow \Delta$

2. All **leafs** are axioms

3. The **nodes** are sequents such that each sequent on the tree follows from the ones immediately preceding it by one of the rules.

Proof Trees

Remark

The proof search in **GL** as defined by the **decomposition tree** for a given formula A **is not always unique**

We show an **example** on the next slide

Example

A tree-proof in **GL** of the de Morgan Law

$$\longrightarrow \; (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\longrightarrow\Rightarrow)$$

$$\neg(a \cap b) \; \longrightarrow \; (\neg a \cup \neg b)$$

$$\mid (\longrightarrow \cup)$$

$$\neg(a \cap b) \; \longrightarrow \; \neg a, \neg b$$

$$\mid (\longrightarrow \neg)$$

$$b, \neg(a \cap b) \; \longrightarrow \; \neg a$$

$$\mid (\longrightarrow \neg)$$

$$b, a, \neg(a \cap b) \; \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$b, a \longrightarrow \; (a \cap b)$$

$$\bigwedge (\longrightarrow \cap)$$

$$b, a \longrightarrow \; a \qquad\qquad b, a \longrightarrow \; b$$

Example

Here is another tree-proof in **GL** of the de Morgan Law

$$\longrightarrow \; (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\longrightarrow \Rightarrow)$$

$$\neg(a \cap b) \; \longrightarrow \; (\neg a \cup \neg b)$$

$$\mid (\longrightarrow \cup)$$

$$\neg(a \cap b) \; \longrightarrow \; \neg a, \neg b$$

$$\mid (\longrightarrow \neg)$$

$$b, \neg(a \cap b) \; \longrightarrow \; \neg a$$

$$\mid (\neg \longrightarrow)$$

$$b \; \longrightarrow \; \neg a, (a \cap b)$$

$$\bigwedge (\longrightarrow \cap)$$

$$b \longrightarrow \; \neg a, a \qquad\qquad b \longrightarrow \; \neg a, b$$

$$\mid (\longrightarrow \neg) \qquad\qquad \mid (\longrightarrow \neg)$$

$$b, a \longrightarrow \; a \qquad\qquad b, a \longrightarrow \; b$$

Decomposition Trees

The process of **searching for proofs** of a formula A in **GL** consists, as in the **RS** type systems, of building certain trees, called decomposition trees

Their **construction** is similar to the one for **RS** type systems

We take a **root** of a **decomposition tree** T_A of of a formula A a sequent $\longrightarrow A$

For each **node**, if there is a rule of **GL** which conclusion has the same form as **node** sequent, then the **node** has **children** that are **premises** of the rule

If the **node** consists only of a sequent built only out of variables then it **does not** have any children

This is a termination condition for the tree

Decomposition Trees

We **prove** that each formula A generates a **finite set**

$$\mathcal{T}_A$$

of **decomposition trees** such that the following holds

If there exist a tree $T_A \in \mathcal{T}_A$ whose **all leaves** are axioms, then tree T_A constitutes a **proof** of A in **GL**

If **all trees** in $\mathcal{T}_A$ have at **least one** non-axiom leaf, the proof of A **does not exist**

System **GL** Exercises

Exercise

Prove, by constructing proper **decomposition trees** that

$$\nvdash_{\mathbf{GL}} ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

Solution

For some formulas A, their decomposition tree $\mathbf{T}_{\rightarrow A}$ may **not be** unique

Hence we have to construct all possible **decomposition** trees to show that none of them is a **proof**, i.e. to show that each of them has a non axiom leaf.

We construct the decomposition trees for $\longrightarrow A$ as follows

System **GL** Exercises

$\mathbf{T}_{1\rightarrow A}$

$$\longrightarrow ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$\mid (\rightarrow\Rightarrow)$ (*one choice*)

$$(a \Rightarrow b) \;\longrightarrow\; (\neg b \Rightarrow a)$$

$\mid (\rightarrow\Rightarrow)$ (*first of two choices*)

$$\neg b, (a \Rightarrow b) \;\longrightarrow\; a$$

$\mid (\neg \rightarrow)$ (*one choice*)

$$(a \Rightarrow b) \;\longrightarrow\; b, a$$

$\bigwedge (\Rightarrow\longrightarrow)$ (*one choice*)

$$\longrightarrow\; a, b, a \qquad\qquad b \longrightarrow b, a$$

non – axiom *axiom*

The tree contains a **non- axiom** leaf, hence it is **not a proof**

We have one more tree to construct

System **GL** Exercises

$\mathbf{T}_{\mathbf{2}\rightarrow A}$

$$\longrightarrow ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow) \ (\textit{one choice})$$

$$(a \Rightarrow b) \ \longrightarrow \ (\neg b \Rightarrow a)$$

$$\bigwedge(\Rightarrow\longrightarrow) \ (\textit{second choice})$$

$$\longrightarrow \ (\neg b \Rightarrow a), a$$

$$\mid (\longrightarrow\Rightarrow) \ (\textit{one choice})$$

$$\neg b \longrightarrow \ a, a$$

$$\mid (\neg \rightarrow) \ (\textit{one choice})$$

$$\longrightarrow \ b, a, a$$

non – axiom

$$b \longrightarrow \ (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow) \ (\textit{one choice})$$

$$b, \neg b \longrightarrow \ a$$

$$\mid (\neg \rightarrow) \ (\textit{one choice})$$

$$b \ \longrightarrow \ b, a$$

axiom

All possible trees end with a non-axiom leaf. It proves that

$$\nvdash_{\mathbf{GL}} ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

System **GL** Exercises

Does the tree below constitute a proof in **GL** ? Justify your answer

$\mathbf{T}_{\rightarrow A}$

$\longrightarrow \neg\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$

$\mid (\rightarrow \neg)$

$\neg((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a)) \longrightarrow$

$\mid (\neg \rightarrow)$

$\longrightarrow ((\neg a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$

$\mid (\rightarrow\Rightarrow)$

$(\neg a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$

$\mid (\rightarrow\Rightarrow)$

$(\neg a \Rightarrow b), \neg b \longrightarrow a$

$\mid (\neg \rightarrow)$

$(\neg a \Rightarrow b) \longrightarrow b, a$

$\bigwedge (\Rightarrow\longrightarrow)$

$\longrightarrow \neg a, b, a$ $\qquad\qquad$ $b \longrightarrow b, a$

$\mid (\rightarrow \neg)$ $\qquad\qquad$ *axiom*

$a \longrightarrow b, a$

axiom

System **GL** Exercises

Solution

The tree $\mathbf{T}_{\rightarrow A}$ is **not a proof** in **GL** because a rule corresponding to the decomposition step below **does not exists** in **GL**

$$(\neg a \Rightarrow b), \neg b \longrightarrow a$$

$$\mid (\neg \rightarrow)$$

$$(\neg a \Rightarrow b) \longrightarrow b, a$$

The tree $\mathbf{T}_{\rightarrow A}$ **is** a proof is some system **GL1** that has all the rules of **GL** except of its $(\neg \rightarrow)$ rule:

$$(\neg \rightarrow) \quad \frac{\Gamma', \Gamma \ \longrightarrow \ \Delta, \ A, \ \Delta'}{\Gamma', \ \neg A, \ \Gamma \ \longrightarrow \ \Delta, \Delta'}$$

This **GL** rule has to be replaced in **GL1** by the rule:

$$(\neg \rightarrow)_1 \ \frac{\Gamma, \Gamma' \ \longrightarrow \ \Delta, A, \Delta'}{\Gamma, \neg A, \Gamma' \ \longrightarrow \ \Delta, \Delta'}$$

Exercises

Exercise 1

Write all possible proofs of

$$(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

Exercise 2

Find a formula which has a unique decomposition tree

Exercise 3

Describe for which kind of formulas the decomposition tree is unique

GL Soundness and Completeness

GL Strong Soundness

The system **GL** admits a constructive proof of the **Completeness Theorem**, similar to completeness proofs for **RS** type proof systems

The completeness proof relays on the **strong soundness** property of the inference rules

We prove the **strong soundness** property of the proof system **GL**

GL Strong Soundness

The strong soundness of the **rules** of inference means that if at least one of premisses of a rule is **false**, the conclusion of the rule is also **false**

Hence given a sequent $\Gamma \longrightarrow \Delta \in SQ$, such that its **decomposition tree** $\mathbf{T}_{\Gamma \longrightarrow \Delta}$ has a **branch** ending with a non-axiom **leaf**

It means that **any** truth assignment v that makes this non-axiom leaf bf false also **falsifies** all sequents on that branch

Hence v **falsifies** the sequent $\Gamma \longrightarrow \Delta$

Counter Model

In particular, given a sequent

$$\longrightarrow A$$

and its **decomposition tree**

$$\mathbf{T}_{\longrightarrow A}$$

any v, that **falsifies** its non-axiom **leaf** is a **counter-model** for the formula A

We call such v a **counter model** determined by the **decomposition tree**

Counter Model Theorem

We have hence proved the following

Counter Model Theorem

Given a sequent $\Gamma \longrightarrow \Delta$, such that its **decomposition tree** $\mathbf{T}_{\Gamma \longrightarrow \Delta}$ contains a non- axiom leaf L_A
Any truth assignment v that **falsifies** the non-axiom leaf L_A is a **counter model** for $\Gamma \longrightarrow \Delta$

In particular, given a formula $A \in \mathcal{F}$, and its **decomposition tree** $\mathbf{T}_A$ with a non-axiom leaf, this leaf let us **define** a counter-model for A **determined** by the decomposition tree $\mathbf{T}_A$

Exercise

Exercise

We know that the system **GL** is **strongly sound**

Prove, by constructing a **counter-model** determined by a proper **decomposition tree** that

$$\not\models ((b \Rightarrow a) \Rightarrow (\neg b \Rightarrow a))$$

We construct the decomposition tree for the formula $A = ((b \Rightarrow a) \Rightarrow (\neg b \Rightarrow a))$ as follows

Exercise

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow ((b \Rightarrow a) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\rightarrow\Rightarrow)$$

$$(b \Rightarrow a) \;\longrightarrow\; (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg b, (b \Rightarrow a) \;\longrightarrow\; a$$

$$\mid (\neg \rightarrow)$$

$$(b \Rightarrow a) \;\longrightarrow\; b, a$$

$$\bigwedge (\Rightarrow\longrightarrow)$$

$$\longrightarrow\; b, b, a \qquad\qquad a \longrightarrow b, a$$

non – axiom   *axiom*

Exercise

The non-axiom leaf L_A we want to **falsify** is

$$\longrightarrow\ b, b, a$$

Let $v : VAR \longrightarrow \{T, F\}$ be a truth assignment

By definition of semantic for sequents we have that

$v^*(\longrightarrow b, b, a) = (T \Rightarrow v(b) \cup v(b) \cup v(a))$

Hence $v^*(\longrightarrow b, b, a) = F$ if and only if

$(T \Rightarrow v(b) \cup v(b) \cup v(a)) = F$ if and only if

$v(b) = v(a) = F$

The **counter model** determined by the $\mathbf{T}_{\rightarrow A}$ is any $v : VAR \longrightarrow \{T, F\}$ such that

$$v(b) = v(a) = F$$

GL Completeness

Our goal now is to prove the **Completeness Theorem** for **GL**

Completeness Theorem
For any formula $A \in \mathcal{F}$,

$$\vdash_{\mathbf{GL}} A \qquad \text{if and only if} \qquad \models A$$

Moreover

For any sequent $\Gamma \longrightarrow \Delta \in SQ$,

$$\vdash_{\mathbf{GL}} \Gamma \longrightarrow \Delta \qquad \text{if and only if} \qquad \models \Gamma \longrightarrow \Delta$$

GL Completeness

Proof

We have already proved the **Soundness Theorem**, so we only need to prove the implication:

$$\text{if } \models A \text{ then } \vdash_{\mathbf{GL}} A$$

We **prove** instead of the logically equivalent opposite implication:

$$\text{if } \nvdash_{\mathbf{GL}} A \text{ then } \not\models A$$

GL Completeness

Assume $\nvdash_{\mathbf{GL}} \ A$, i.e. $\nvdash_{\mathbf{GL}} \longrightarrow A$

Let $\mathcal{T}_A$ be a set of **all** decomposition trees of $\longrightarrow A$

As $\nvdash_{\mathbf{GL}} \longrightarrow A$ each tree $\mathbf{T}_{\rightarrow A}$ in the set $\mathcal{T}_A$ has a non-axiom leaf. We choose an arbitrary $\mathbf{T}_{\rightarrow A} \in \mathcal{T}_A$

Let $L_A = \Gamma^{'} \longrightarrow \Delta^{'}$ be a non-axiom leaf of $\mathbf{T}_{\rightarrow A}$

We **define** a truth assignment $v : VAR \longrightarrow \{T, F\}$ which **falsifies** $L_A = \Gamma^{'} \longrightarrow \Delta^{'}$ as follows

$$v(a) = \begin{cases} T & \text{if } a \text{ appears in } \Gamma' \\ F & \text{if } a \text{ appears in } \Delta' \\ \textit{any value} & \text{if } a \text{ does not appear in } \Gamma' \rightarrow \Delta' \end{cases}$$

By **Counter Model Theorem**

$$\not\models \ A$$