cse371/math371
LOGIC

Professor Anita Wasilewska

LECTURE 7

Chapter 7
Introduction to Intuitionistic and Modal Logics

PART 1: **Intuitionictic Logic: Philosophical Motivation**

Intuitionictic Logic: Philosophical Motivation

Intuitionistic logic has developed as a result of certain philosophical views on the foundation of mathematics, known as intuitionism

Intuitionism was originated by L. E. J. Brouwer in 1908

The first Hilbert style formalization of the intuitionistic logic, formulated as a proof system, is due to A. Heyting (1930)

We **present** a Hilbert style proof system I that is equivalent to the Heyting's original formalization

We also **discuss** the relationship between intuitionistic and classical logic.

Intuitionictic Logic: Philosophical Motivation

There have been several successful attempts at creating **semantics** for the intuitionistic logic. The most **recent** called Kripke models were defined by Kripke in 1964

The **first** intuitionistic semantics was defined in a form of **pseudo-Boolean** algebras by McKinsey and Tarski in years 1944 - 1946
Their **algebraic** approach to intuitionistic and classical semantics was followed by many authors and developed into a new field of **Algebraic Logic**

The pseudo- Boolean algebras are called also Heyting algebras to memorize his **first** accepted formalization of the intuitionistic logic as a proof system

Intuitionictic Logic: Philosophical Motivation

An uniform presentation of **algebraic models** for classical, intuitionistic and modal logics S4, S5 was first given in a now classic **algebraic logic** book:

"Mathematics of Metamathematics", Rasiowa, Sikorski (1964)

The main **goal** of this chapter is to give a presentation of the intuitionistic logic formulated as Hilbert and Gentzen proof systems

We also discuss its **algebraic** semantics and the fundamental theorems that establish the relationship between classical and intuitionistic propositional logics

Intuitionictic Logic: Philosophical Motivation

Intuitionists' view-point on the **meaning** of the basic logical and set theoretical concepts used in mathematics **is different** from that of most mathematicians use in their research

The basic **difference** between the intuitionist and classical mathematician lies in the **interpretation** of the word exists For example, let $A(x)$ be a statement in the arithmetic of natural numbers. For the mathematicians the sentence $\exists x A(x)$ is **true** if it is a theorem of arithmetic

If a mathematician **proves** sentence $\exists x A(x)$ this **does not always** mean that he is able to indicate a method of construction of a natural number n such that $A(n)$ holds

Intuitionictic Logic: Philosophical Motivation

Moreover, the mathematician often obtains the **proof** of the existential sentence $\exists x A(x)$ by **proving** first a sentence

$$\neg \forall x \neg A(x)$$

Next he makes use of a classical tautology

$$(\neg \forall x \neg A(x)) \Rightarrow \exists x A(x))$$

By applying Modus Ponens he obtains the **proof** of the existential sentence

$$\exists x A(x)$$

For the intuitionist such method is **not acceptable**, for it **does not** give any method of **constructing** a number n such that $A(n)$ holds

Intuitionictic Logic: Philosophical Motivation

For this reason the intuitionist **do not accept** the classical tautology

$$(\neg \forall x \, \neg A(x)) \Rightarrow \exists x A(x))$$

as intuitionistic tautology, or as as an intuitionistically **provable** sentence

Intuitionictic Logic: Philosophical Motivation

Let us denote by $\vdash_I A$ and $\models_I A$ the fact that A is intuitionistically provable and that A is intuitionistic tautology, respectively

The **proof system** I for the intuitionistic logic has hence to be such that

$$\nvdash_I \ (\neg\forall x\, \neg A(x)) \Rightarrow \exists x A(x))$$

The intuitionistic semantics I has to be such that

$$\not\models_I \ (\neg\forall x\, \neg A(x)) \Rightarrow \exists x A(x))$$

The above means also that intuitionists interpret **differently** the meaning of propositional connectives

Intuitionistic implication

The intuitionistic implication $(A \Rightarrow B)$ is considered by to be **true** if there exists a method by which a proof of B can be **deduced** from the proof of A

In the case of the implication

$$i(\neg \forall x \, \neg A(x)) \Rightarrow \exists x A(x))$$

there is no general method which, from a proof of the sentence

$$(\neg \forall x \, \neg A(x))$$

permits us to **obtain** an intuitionistic proof of the sentence

$$\exists x A(x)$$

Intuitionictic Logic: Philosophical Motivation

Intuitionistic negation

The sentence $\neg A$ is considered intuitionistically **true** if the **acceptance** of the sentence A leads to **absurdity**

As a result of above understanding of negation and implication we have that in the intuitionistic proof system I

$$\vdash_I (A \Rightarrow \neg\neg A) \quad \text{but} \quad \nvdash_I (\neg\neg A \Rightarrow A)$$

Consequently, the intuitionistic semantics I has to be such that

$$\models_I (A \Rightarrow \neg\neg A) \quad \text{and} \quad \not\models_I (\neg\neg A \Rightarrow A)$$

Intuitionistic disjunction

The intuitionist regards a **disjunction** $(A \cup B)$ as **true** if one of the sentences A, B is **true** and there is a method by which it is possible to find out which of them is **true**

As a consequence a classical law of **excluded middle**

$$(A \cup \neg A)$$

is not acceptable by the intuitionists

This means that the the intuitionistic proof system I must be such that

$$\nvdash_I (A \cup \neg A)$$

and the intuitionistic semantics I has to be such that

$$\not\models_I (A \cup \neg A)$$

Chapter 7
Introduction to Intuitionistic and Modal Logics

PART 2: Intuitionistic Proof System P*I*,

Algebraic Semantics and Completeness Theorem

Intuitionistic Proof System I

We define now a Hilbert style **proof system** I with a set of axioms that is due to Rasiowa (1959). We adopted this axiomatization for two reasons

First reason is that it is the most natural and appropriate set of axioms to carry the the algebraic proof of the **completeness theorem**

Second reason is that they clearly describe the main **difference** between intuitionistic and classical logic
Namely, by **adding** to I the only one more axiom

$$(A \cup \neg A)$$

we get a **complete** formalization for classical logic

Intuitionistic Proof System I

Here are the components if the proof system I

Language

We adopt a propositional language

$$\mathcal{L} = \mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}}$$

with the set of formulas $\mathcal{F}$

Axioms

A1 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$

A2 $(A \Rightarrow (A \cup B))$

A3 $(B \Rightarrow (A \cup B))$

A4 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$

A5 $((A \cap B) \Rightarrow A)$

A6 $((A \cap B) \Rightarrow B)$

A7 $((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B))))$

Intuitionistic Proof System I

A7 $((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B)))$

A8 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$

A9 $(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)),$

A10 $(A \cap \neg A) \Rightarrow B),$

A11 $((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A),$

where A, B, C are any formulas in $\mathcal{L}$

Rules of inference

We adopt the Modus Ponens

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

as the **only** rule of inference

Intuitionistic Proof System I

A proof system

$$\mathbf{I} = (\ \mathcal{L}, \mathcal{F}\ A1 - A11,\ (MP)\ )$$

for axioms A1 - A11 defined above is called a Hilbert style **formalization** for intuitionistic propositional logic

We introduce, as usual, the notion of a **formal proof** in I and denote by

$$\vdash_I A$$

the fact that a formula A has a formal **proof** in I or that A v is **provable** in I

Algebraic Semantics and Completeness Theorem

Algebraic Semantics

We present now a short version of Tarski, Rasiowa, and Sikorski **psedo-Boolean** algebra semantics

We also discuss the algebraic **completeness theorem** for the intuitionistic propositional logic

We leave the **Kripke semantics** for the reader to **explore** from other, multiple sources

Algebraic Semantics

Here are some **basic** definitions

Relatively Pseudo-Complemented Lattice (Birkhoff, 1935)

A lattice

$$(B, \cap, \cup)$$

is said to be relatively pseudo-complemented if and only if for any elements $a, b \in B$, there exists the **greatest** element c, such that

$$a \cap c \leq b$$

Such greatest element c is denoted by $a \Rightarrow b$ and called the **pseudo-complement** of a **relative** to b

Algebraic Semantics

Directly from definition we have that

$$(*) \quad x \leq a \Rightarrow b \quad \text{if and only if} \quad a \cap x \leq b \quad \text{for all} \quad x, a, b \in B$$

This equation $(*)$ can serve as the **definition** of the relative pseudo-complement $a \Rightarrow b$

Algebraic Semantics

Fact

Every relatively pseudo-complemented lattice $(B, \cap, \cup)$ has the **greatest** element, called a unit element and denoted by 1

Proof

Observe that $a \cap x \leq a$ for all $x, a \in B$

By $(*)$ we have that $x \leq a \Rightarrow a$ for all $x \in B$

This means that $a \Rightarrow a$ is the greatest element in the lattice $(B, \cap, \cup)$. We write it as

$$a \Rightarrow a = 1$$

Algebraic Semantics

Definition

An abstract algebra

$$\mathcal{B} = (B,\ 1,\ \Rightarrow,\ \cap,\ \cup)$$

is said to be a **relatively pseudo-complemented lattice** if and only if $(B, \cap, \cup)$ is a relatively pseudo-complemented lattice with the relative pseudo-complement $\Rightarrow$ defined by the equation

$(*)$ $x \leq a \Rightarrow b$ if and only if $a \cap x \leq b$ for all $x, a, b \in B$

and with the unit element 1

Relatively Pseudo-complemented Set Lattices

Consider a **topological** space X with an interior operation I
Let $\mathcal{G}(X)$ be the class of all open subsets of X and
$\mathcal{G}^*(X)$ be the class of all both dense and open subsets of X
Then the algebras

$$(\mathcal{G}(X),\ X,\ \cup,\ \cap, \Rightarrow),\quad (\mathcal{G}^*(X),\ X,\ \cup,\ \cap, \Rightarrow)$$

where $\cup$, $\cap$ are set-theoretical operations of union, intersection, and $\Rightarrow$ is defined by

$$Y \Rightarrow Z = I(X - Y) \cup Z$$

are relatively pseudo-complemented lattices

Algebraic Semantics

Clearly, all sub algebras of these algebras are also **relatively pseudo-complemented lattices**

They are typical examples of relatively pseudo complemented lattices

Algebraic Semantics

Pseudo - Boolean Algebra (Heyting Algebra)

An algebra

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \ \cap,\ \cup, \neg)$$

is said to be a **pseudo - Boolean** algebra if and only if

$$(B,\ 1,\ \Rightarrow,\ \ \cap,\ \cup)$$

is a relatively pseudo-complemented lattice in which a zero element 0 exists and $\neg$ is a one argument operation defined as follows

$$\neg a = a \Rightarrow 0$$

Algebraic Semantics

The operation $\neg$ defined as

$$\neg a = a \Rightarrow 0$$

is called a **pseudo-complementation**

The **pseudo - Boolean** algebras are also called **Heyting** algebras to stress their connection to the intuitionistic logic

Algebraic Semantics

Let X be **topological** space with an interior operation I

Let $\mathcal{G}(X)$ be the class of all open subsets of X

Then

$$(\mathcal{G}(X),\ X,\ \emptyset,\ \cup,\ \cap, \Rightarrow,\ \neg)$$

where $\cup$, $\cap$ are set-theoretical operations of union, intersection, and $\Rightarrow$ is defined by

$$Y \Rightarrow Z = I(X - Y) \cup Z$$

and $\neg$ is defined as

$$\neg Y = Y \Rightarrow \emptyset = I(X - Y), \quad \text{for all } Y \subseteq X$$

is a **pseudo - Boolean** algebra

Every sub algebra of $\mathcal{G}(X)$ is also a pseudo-Boolean algebra

They are called **pseudo-fields of sets**

Algebraic Semantics

The following theorem states that pseudo-fields are typical **examples** of pseudo - Boolean algebras.

The theorems of this type are often called **Stone Representation Theorems** to remember an American mathematician H. M. Stone

Stone was one of the **first** to initiate the investigations of **relationship** between logic and general topology in the article

"The Theory of Representations for Boolean Algebras",

Trans. of the Amer.Math, Soc 40, 1936

Representation Theorem (McKinsey, Tarski, 1946)

For every **pseudo - Boolean** algebra

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg)$$

there exists a monomorphism h of $\mathcal{B}$ into a **pseudo-field** $\mathcal{G}(X)$ of all open subsets of a compact topological T_0 space X

Intuitionistic Algebraic Model

We say that a formula A is an **intuitionistic tautology**

if and only if

any **pseudo-Boolean** algebra $\mathcal{B}$ is a **model** for A

This kind of **models** because their connection to abstract algebras are called **algebraic models**

We put it formally as follows.

Intuitionistic Algebraic Model

Let A be a formula of the language $\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}}$ and let

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg)$$

be a **pseudo - Boolean** algebra

We say that the algebra $\mathcal{B}$ is a **model** for the formula A and denote it by

$$\mathcal{B} \models A$$

if and only if $v^*(A) = 1$ holds for all variables assignments

$$v : VAR \longrightarrow B$$

Intuitionistic Tautology

The formula A is an **intuitionistic tautology** and is denoted by

$$\models_I A$$

if and only if

$$\mathcal{B} \models A \quad \text{for all pseudo-Boolean algebras } \mathcal{B}$$

Intuitionistic Tautology

In **Algebraic Logic** the notion of tautology is often defined using a notion

"a formula A is **valid** in an algebra $\mathcal{B}$"

It is formally defined as follows

Intuitionistic Tautology

Definition

A formula A **is valid** in a pseudo-Boolean algebra

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg)$$

if and only if $v^*(A) = 1$ holds for all variables assignments $v : VAR \longrightarrow B$

Directly from definitions we get the following Fact

Intuitionistic Tautology

Fact

For any formula A,

$\models_I A$ if and only if A is **valid**

in all pseudo-Boolean algebras $\mathcal{B}$

The **Fact** is often used as an equivalent **definition** of the intuitionistic tautology

Intuitionistic Completeness

We write now $\vdash_I A$ to denote **any** proof system for the intuitionistic propositional logic, and in particular the Rasiowa (1959) proof system we have defined

Intuitionistic Completeness Theorem (Mostowski 1948)

For any formula A of $\mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg\}}$,

$$\vdash_I A \quad \text{if and only if} \quad \models_I A$$

The intuitionistic completeness theorem follows directly from the general **algebraic completeness theorem** that combines results of of Mostowski (1958), Rasiowa (1951) and Rasiowa-Sikorski (1957)

Algebraic Completeness Theorem

For any formula A he following conditions are equivalent

(i) $\vdash_I A$

(ii) $\models_I A$

(iii) A is **valid** in every pseudo-Boolean algebra

$$(\mathcal{G}(X),\ X,\ \emptyset,\ \cup,\ \cap, \Rightarrow,\ \neg)$$

of open subsets of any topological space X

(iv) A is **valid** in every pseudo-Boolean algebra $\mathcal{B}$ with at most 2^{2^r} elements, where r is the number of all sub formulas of A

Moreover, each of the conditions **(i)** - **(iv)** is equivalent to the following one.

(v) A is **valid** in the pseudo-Boolean algebra $(\mathcal{G}(X),\ X,\ \emptyset,\ \cup,\ \cap, \Rightarrow,\ \neg)$ of open subsets of a dense-in -itself metric space $X \neq \emptyset$ (in particular of an n-dimensional Euclidean space X)

Chapter 7
Introduction to Intuitionistic and Modal Logics

PART 3: Intuitionistic Tautologies and Connection with Classical Tautologies

Intuitionistic Tautologies

Here are some important **basic** classical tautologies that are also intuitionistic **tautologies**

$(A \Rightarrow A)$

$(A \Rightarrow (B \Rightarrow A))$

$(A \Rightarrow (B \Rightarrow (A \cap B)))$

$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$

$(A \Rightarrow \neg\neg A)$

$\neg(A \cap \neg A)$

$((\neg A \cup B) \Rightarrow (A \Rightarrow B))$

Of course, all of logical axioms A1 - A11 of the proof system **I** are also classical and intuitionistic tautologies

Intuitionistic Tautologies

Here are some **more** of important classical tautologies that **are** intuitionistic **tautologies**

$((\neg A \cup B) \Rightarrow (A \Rightarrow B))$

$(\neg(A \cup B) \Rightarrow (\neg A \cap \neg B))$

$((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$

$((\neg A \cup \neg B) \Rightarrow \neg(A \cap B))$

$((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

$((A \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg A))$

$(\neg\neg\neg A \Rightarrow \neg A)$

$(\neg A \Rightarrow \neg\neg\neg A)$

$(\neg\neg(A \Rightarrow B) \Rightarrow (A \Rightarrow \neg\neg B))$

$((C \Rightarrow A) \Rightarrow ((C \Rightarrow (A \Rightarrow B)) \Rightarrow (C \Rightarrow B))$

Intuitionistic Tautologies

Here are some important classical tautologies that **are not** intuitionistic **tautologies**

$(A \cup \neg A)$

$(\neg\neg A \Rightarrow A)$

$((A \Rightarrow B) \Rightarrow (\neg A \cup B))$

$(\neg(A \cap B) \Rightarrow (\neg A \cup \neg B))$

$((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow A))$

$((\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A))$

$(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$

Connection Between Classical and Intuitionistic Logics

Connection Between Classical and Intuitionistic Logics

The first **connection** is quite obvious

It was proved by Rasiowa, Sikorski in 1964 that by adding the axiom

A12 $(A \cup \neg A)$

to the set of of logical axioms A1 - A11 of the proof system I we obtain a proof system C that is **complete** with respect to classical semantics

This proves the following

Theorem 1

Every formula that is intuitionistically derivable is also classically derivable, i.e. the implication

$$\text{If} \quad \vdash_I A \quad \text{then} \quad \vdash_C A$$

holds for any $A \in \mathcal{F}$

Classical and Intuitionistic Logics

We write $\models A$ and $\models_I A$ to denote that A is a classical and intuitionistic tautology, respectively.

As both proof systems I and C are **complete** under respective semantics, we can re-write **Theorem 1** as the following **relationship** between classical and intuitionistic tautologies

Theorem 2

For any formula $A \in \mathcal{F}$,

$$\text{If } \models_I A, \text{ then } \models A$$

The next **relationship** shows how to obtain intuitionistic tautologies from the classical tautologies and vice versa

The following has been proved by Glivenko in 1929 and independently by Tarski in 1938

Theorem 3 (Glivenko, Tarski)

For any formula $A \in \mathcal{F}$,

A is classically provable if and only if $\neg\neg A$ is intuitionistically provable, i.e.

$$\vdash A \quad \text{if and only if} \quad \vdash_I \neg\neg A$$

where we use symbol $\vdash$ for classical provability

Theorem 4 (Tarski, 1938)

For any formula $A \in \mathcal{F}$,

A is a classical tautology if and only if $\neg\neg A$ is an intuitionistic tautology, i.e.

$$\models A \quad \text{if and only if} \quad \models_I \neg\neg A$$

Classical and Intuitionistic Logics

Theorem 5 (Gödel, 1931)

For any formulas $A, B \in \mathcal{F}$,

a formula $(A \Rightarrow \neg B)$ is classically provable if and only if it is intuitionistically provable, i.e.

$$\vdash (A \Rightarrow \neg B) \quad \text{if and only if} \quad \vdash_I (A \Rightarrow \neg B)$$

Theorem 6 (Gödel, 1931)

For any formula $A, B \in \mathcal{F}$,

If A contains **no connectives** except $\cap$ and $\neg$,

then A is classically provable if and only if it is intuitionistically provable, i.e

$$\vdash A \quad \text{if and only if} \quad \vdash_I A$$

Classical and Intuitionistic Logics

By the completeness of classical and intuitionisctic logics we get the following **semantic** version of Gödel' s **Theorems 5, 6**

Theorem 7

A formula $(A \Rightarrow \neg B)$ is a classical tautology if and only if it is an intuitionistic tautology, i.e.

$$\models (A \Rightarrow \neg B) \quad \text{if and only if} \quad \models_I (A \Rightarrow \neg B)$$

Theorem 8

If a formula A contains no connectives except $\cap$ and $\neg$, then

$$\models A \quad \text{if and only if} \quad \models_I A$$

On intuitionistically derivable disjunction

In classical logic it is possible for the disjunction

$$(A \cup B)$$

to be a **tautology** when neither A nor B is a **tautology**

The tautology $(A \cup \neg A)$ is the simplest example

This does not hold for the intuitionistic logic

This fact was **stated** without the proof by Gödel in 1931 and **proved** by Gentzen in 1935 via his proof system **LI** which was discussed shortly in chapter 6 and is covered in detail in this chapter and the next set of slides

On intuitionistically derivable disjunction

The following theorem was announced without proof by Gödel in 1931 and proved by Gentzen in 1935

Theorem 9 (Gödel, Gentzen)

A disjunction $(A \cup B)$ is intuitionistically provable if and only if either A or B is intuitionistically provable i.e.

$$\vdash_I (A \cup B) \quad \text{if and only if} \quad \vdash_I A \ \text{ or } \ \vdash_I B$$

On intuitionistically derivable disjunction

We obtain, via the **Completeness Theorems** the following semantic version of the above (Gödel, Gentzen) Theprem 9

Theorem 10

A disjunction $(A \cup B)$ is intuitionistic tautology if and only if either A or B is intuitionistic tautology, i.e.

$$\models_I (A \cup B) \quad \text{if and only if} \quad \models_I A \quad \text{or} \quad \models_I B$$