cse371/ math371
LOGIC

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LECTURE 7b

Chapter 7
Introduction to Intuitionistic and Modal Logics

PART 5: **Introduction to Modal Logics**

Algebraic Semantics for modal S4 and S5

Introduction to Modal Logics

The **non-classical** logics can be divided in two groups: those that **rival** classical logic and those which **extend it**

The Lukasiewicz, Kleene, and intuitionistic logics are in the **first** group, the modal logics are in the **second** group

The **rival** logics **do not** differ from classical logic in terms of the language employed

The **rival** logics differ in that certain theorems or tautologies of classical logic are rendered **false**, or **not provable** in them

The most notorious example of the **rival** difference of logics based on the same language is the law of excluded middle

$$(A \cup \neg A)$$

This is **provable** in, and is a **tautology** of classical logic

But **is not** provable in, and **is not** tautology of the intuitionistic logic

It also **is not** a tautology under any of the extensional logics semantics we have discussed

Introduction to Modal Logics

Logics which **extend classical** logic sanction all the theorems of classical logic but, generally, supplement it in **two** ways

Firstly, the languages of these non-classical logics are **extensions** of those of classical logic

Secondly, the theorems of these non-classical logics **supplement** those of classical logic

Introduction to Modal Logics

Modal logics are enriched by the addition of two new **connectives** that represent the meaning of expressions "it is necessary that" and " it is possible that"

We use the notation:

I for " it is necessary that" and

C for " it is possible that"

Other notations commonly used are:

∇, N, L for " it is necessary that" and

$\diamond$, P, M for " it is possible that"

Introduction to Modal Logics

The symbols N, L, P, M or alike, are often used in computer science

The symbols ∇ and $\diamond$ were **first** to be used in modal logic literature

The symbols **I, C** come from **algebraic** and **topological** interpretation of modal logics

I corresponds to the topological **interior** of the set and **C** to its **closure**

Introduction to Modal Logics

The **idea** of a modal logic was **first** formulated by an American philosopher, C.I. Lewis in 1918

Lewis has proposed yet another interpretation of lasting consequences, of the logical **implication**

He created a notion of a **modal truth**, which lead to the notion of modal logic

He did it in an attempt to avoid, what some felt, the **paradoxes** of semantics for classical **implication** which accepts as **true** that a false sentence **implies** any sentence

Introduction to Modal Logics

Lewis' notions appeal to **epistemic** considerations and the whole area of modal logics bristles with **philosophical** difficulties and hence the numbers of modal logics have been **created**

Unlike the classical connectives, the modal connectives **do not** admit of truth-functional interpretation, i.e. the modal connectives **do not accept** the extensional semantics

This was the **reason** for which modal logics were **first** developed as proof systems, with intuitive notion of **semantics** expressed by the set of adopted axioms

Introduction to Modal Logics

The **first definition** of modal semantics, and hence the proofs of the **completeness** theorems came some 20 years later

It took yet another 25 years for discovery and development of the **second** and more general approach to the modal semantics

These are the two **established** ways to **interpret** modal connectives, i.e. to **define** the modal semantics

Introduction to Modal Logics

The historically, the **first** modal **semantics** is due to Mc Kinsey and Tarski (1948)

It is a **topological semantics** that provides a powerful mathematical **interpretation** of some of modal logics, namely modal S4 and S5

It connects the **modal** notion of necessity with the **topological** notion of the interior of a set, and the **modal** notion of possibility with the notion of the closure of a set

Introduction to Modal Logics

Our **choice** of symbols **I** and **C** for necessity and possibility **connectives**, respectively comes from their **topological** interpretation

The **topological** interpretation mathematically powerful as it is, is less **universal** in providing models for **other** modal logics

Introduction to Modal Logics

The most recent and the most **general** modal semantics is due to Kripke (1964) and uses the notion of possible worlds

Roughly, we say that the formula **C***A* is **true** if *A* is **true** in **some** possible world, called **actual world**

The formula **I***A* is **true** if *A* is **true** in **every** possible world

We present here a short version of the **topological** semantics in a form of **algebraic models**

We leave the **Kripke semantics** for the reader to **explore** from other, multiple sources

Introduction to Modal Logics

As we have already mentioned, modal logics were first **developed**, as was the intuitionistic logic, in a **form** of proof systems only

First several Hilbert style formalizations (proof systems) for **modal** logics were published by Lewis and Langford in in 1932

They presented a formalization for **two** modal logics, which they called S1 and S2 and outlined **three** other proof systems, called S3, S4, and S5

Since then **hundreds** of modal logics have been and still are **created** and investidated

Some **standard**, important and vidlely used books on Modal ooks Logics were written by the following authors

Hughes and Cresswell (1969) for **philosophical** motivation for various modal logics and the intuitionistic logic

Bowen (1979) for a detailed and uniform study of **Kripke models** for modal logics

Segeberg (1971) for excellent modal logics **classification**

Fitting (1983) for extended and uniform studies of **automated proof systems** and methods for **classes** of modal logics

Hilbert Style Modal Proof Systems

Hilbert Style Modal Proof Systems

We present now Hilbert style formalization for S4 and S5 logics that are due to Mc Kinsey and Tarski (1948), and Rasiowa and Sikorski (1964)

We also discuss the **relationship** between S4 and S5 , and between the intuitionistic logic and S4 modal logic, as was first observed by Gödel

The formalizations stress the **connection** between S4, S5 and topological spaces which constitute **models** for them

Modal Language

We **add** two extra one argument connectives **I** and **C** to the propositional language $\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}}$, i.e. we adopt

$$\mathcal{L} = \mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg,\mathbf{I},\mathbf{C}\}}$$

as the modal language and we read formulas $\mathbf{I}A$, $\mathbf{C}A$ as necessary A and possible A, respectively

Modal Language

The **Modal Language**

$$\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg,\mathbf{I},\mathbf{C}\}}$$

is **common** to all modal logics

Modal logics differ on a **choice** of axioms and rules of inference, when studied as proof systems and on a **choice** of respective semantics

McKinsey, Tarski Proof Systems

As modal logics extend the classical logic, any modal logic contains **two groups** of axioms: classical and modal

McKinsey, Tarski **Proof System** (1948)

Classical Axioms

We adopt as classical axioms any **complete** set of axioms under classical semantics

Modal Axioms

M1 $(\mathbf{I}A \Rightarrow A)$

M2 $(\mathbf{I}(A \Rightarrow B) \Rightarrow (\mathbf{I}A \Rightarrow \mathbf{I}B))$

M3 $(\mathbf{I}A \Rightarrow \mathbf{II}A)$

M4 $(\mathbf{C}A \Rightarrow \mathbf{IC}A)$

Modal S4 and S5

Rules of inference

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}, \quad \text{and} \quad (\mathbf{I})\ \frac{A}{\mathbf{I}A}$$

The modal rule $(\mathbf{I})$ was introduced by Gödel and is referred to as a necessitation rule

We define **modal** proof systems S4 and S5 as follows

$$S4 = (\ \mathcal{L},\ \mathcal{F},\ \text{Classical Axioms},\ M1 - M3,\ (MP),\ (\mathbf{I})\ )$$

$$S5 = (\ \mathcal{L},\ \mathcal{F},\ \text{Classical Axioms},\ M1 - M4,\ (MP),\ (\mathbf{I})\ )$$

Modal S4 and S5

Observe that the axioms of S5 **extend** the axioms of S4 and both system **share** the same inference rules, hence we immediately have the following fact

Fact For any formula $A \in \mathcal{F}$,

$$\text{if} \quad \vdash_{S4} A, \quad \text{then} \quad \vdash_{S5} A$$

Rasiowa, Sikorski **Modal Proof System** (1964)

It is often the case, as it is for S4 and S5, that **modal connectives** are definable by each other and are defined as follows

$$\mathbf{I}A = \neg\mathbf{C}\neg A, \quad \text{and} \quad \mathbf{C}A = \neg\mathbf{I}\neg A$$

Language

We hence assume now that the language $\mathcal{L}$ of Rasiowa, Sikorski modal proof systems contains only **one** modal connective and we **choose** it to be $\mathbf{I}$ and adopt the following language

$$\mathcal{L} = \mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg,\mathbf{I}\}}$$

Rasiowa, Sikorski (1964) **Axioms**

There are, as before, **two groups** of axioms: Classical and Modal Axioms

Classical Axioms

We **adopt** as classical axioms any **complete** set of axioms under classical semantics

Modal Axioms

R1 $((\mathbf{I}A \cap \mathbf{I}B) \Rightarrow \mathbf{I}(A \cap B))$

R2 $(\mathbf{I}A \Rightarrow A)$

R3 $(\mathbf{I}A \Rightarrow \mathbf{II}A)$

R4 $\mathbf{I}(A \cup \neg A)$

R5 $(\neg\mathbf{I}\neg A \Rightarrow \mathbf{I}\neg\mathbf{I}\neg A)$

Modal RS4 and RS5

Rules of inference

We adopt the Modus Ponens and an additional rule (R$\mathbf{I}$)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B} \qquad \text{and} \qquad (R\mathbf{I})\ \frac{(A \Rightarrow B)}{(\mathbf{I}A \Rightarrow \mathbf{I}B)}$$

We define modal proof systems $RS4$ and $RS5$ as follows

$$RS4 = (\ \mathcal{L},\ \mathcal{F},\ \text{Classical Axioms},\ R1 - R4,\ (MP),\ (R\mathbf{I})\ )$$

$$RS5 = (\ \mathcal{L},\ \mathcal{F},\ \text{Classical Axioms},\ R1 - R5,\ (MP),\ (R\mathbf{I})\ )$$

Modal RS4 and RS5

Observe that the axioms of RS5 **extend** the axioms of RS4 and both systems **share** the same inference rules, hence we have immediately the following fact

Fact For any formula $A \in \mathcal{F}$,

$$\text{if} \quad \vdash_{RS4} A, \quad \text{then} \quad \vdash_{RS5} A$$

Algebraic Semantics for S4 and S5

Algebraic Semantics for S4 and S5

The McKinsey, Tarski proof systems S4, S5 and the Rasiowa, Sikorski proof systems RS4, RS5 are **complete** with respect to both semantics; the topological **semantics** and the Kripke **semantics**

We shortly discuss the topological semantics, and corresponding **algebraic completeness** theorems

We leave the Kripke semantics for the reader to **explore** from other, multiple sources

Algebraic Semantics for S4 and S5

The investigation of relationship between **topology** and modal logics was initiated by McKinsey in 1941

It continued by McKinsey and Tarski in years 1944 - 1948

It culminated in creation of their algebraic semantics and consequently developed into a field of **Algebraic Logic**

Algebraic Semantics for S4 and S5

The **algebraic** approach to logic is presented in detail in the already classic **algebraic logic** books:

"Mathematics of Metamathematics", Rasiowa, Sikorski (1964),
"An Algebraic Approach to Non-Classical Logics", Rasiowa (1974)

We want to point out that the **first idea** of a connection between **modal** propositional logic and **topology** is due to Tang Tsao -Chen, (1938) and Dugunji (1940)

Algebraic Semantics for S4 and S5

Here are some basic definitions

Boolean Algebra

An abstract algebra $\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg)$ is said to be a **Boolean algebra** if it is a distributive lattice and every element $a \in B$ has a complement $\neg a \in B$

Topological Boolean algebra

By a topological Boolean algebra we mean an abstract algebra

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap, \cup, \neg,\ I)$$

where $(B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg)$ is a **Boolean algebra** and the following conditions hold for any $a, b \in B$

$$I(a \cap b) = Ia \cap Ib, \quad Ia \cap a = Ia, \quad IIa = Ia, \text{ and } I1 = 1$$

Algebraic Semantics for S4 and S5

The element Ia is called a **interior** of a

The element $\neg I \neg a$ is called a **closure** of a and will be **denoted** by Ca

Thus the operations I and C are such that

$$Ca = \neg I \neg a \quad \text{and} \quad Ia = \neg C \neg a$$

In this case we write the **topological Boolean algebra** as

$$\mathcal{B} = (B, \ 1, \ 0, \ \Rightarrow, \ \cap, \ \cup, \neg, \ I, \ C \ )$$

It is easy to prove that in in any topological Boolean algebra the following conditions hold for any $a, b \in B$

$$C(a \cup b) = Ca \cup Cb, \quad Ca \cup a = Ca, \quad CCa = Ca \ \text{and} \ C0 = 0$$

Algebraic Semantics for S4 and S5

Example

Let X be a topological space with an interior operation I

Then the family $\mathcal{P}(X)$ of all subsets of X is a **topological Boolean algebra** with $1 = X$, with

the operation $\Rightarrow$ defined by the formula

$$Y \Rightarrow Z = (X - Y) \cup Z \text{ for all subsets } Y, Z \text{ of } X$$

and with set-theoretical operations of union, intersection, complementation, and the interior operation I

Every sub algebra of this algebra is a **topological Boolean** algebra, called a topological field of sets or, more precisely, a **topological field** of subsets of X

Algebraic Semantics for S4 and S5

Given a topological Boolean algebra

$$(B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg)$$

The element $a \in B$ is said to be **open** (**closed**)
if $a = Ia$ ($a = Ca$)

Clopen Topological Boolean Algebra

A topological Boolean algebra

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg,\ I,\ C\ )$$

such that every **open** element is **closed** and every **closed** element is **open**, i.e. such that for any $a \in B$

$$CIa = Ia \quad \text{and} \quad ICa = Ca$$

is called a **clopen** topological Boolean algebra

S4, S5 Tautology

We loosely say that a formula A is a modal $S4$ **tautology** if and only if any topological Boolean algebra is a **model** for A

We say that A is a modal $S5$ **tautology** if and only if any **clopen** topological Boolean algebra is a **model** for A

We put it formally as follows

Modal Algebraic Model

Modal Algebraic Model

For any formula A of a modal language $\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg,\mathbf{I},\mathbf{C}\}}$ and for any topological Boolean algebra

$$\mathcal{B} = (B,\ 1,\ 0,\ \Rightarrow,\ \cap,\ \cup, \neg,\ I,\ C)$$

the algebra $\mathcal{B}$ is a **model** for the formula A and is denote by

$$\mathcal{B} \models A$$

if and only if $v^*(A) = 1$ holds for all variables assignments $v : VAR \longrightarrow B$

S4, S5 Tautology

Definition of $S4$ **Tautology**

A formula A is a modal $S4$ **tautology** and is denoted by

$$\models_{S4} A$$

if and only if for all **topological Boolean** algebras $\mathcal{B}$ we have that

$$\mathcal{B} \models A$$

Definition of $S5$ **Tautology**

A formula A is a modal $S5$ **tautology** and is denoted by

$$\models_{S5} A$$

if and only if for all **clopen** topological Boolean algebras $\mathcal{B}$ we have that

$$\mathcal{B} \models A$$

S4, S5 Completeness Theorem

We write $\vdash_{S4}$ A and $\vdash_{S5}$ A do denote **provability any** proof system for modal S4, S5 logics and in particular the proof systems defined here

Completeness Theorem

For any formula A of the modal language $\mathcal{L}_{\{\cup, \cap, \Rightarrow, \neg, \mathbf{I}, \mathbf{C}\}}$

$$\vdash_{S4} A \quad \text{if and only if} \quad \models_{S4} A$$

$$\vdash_{S5} A \quad \text{if and only if} \quad \models_{S5} A$$

The completeness for $S4$, $S4$ follows directly from the following general Algebraic Completeness Theorems

S4 Algebraic Completeness Theorem

For any formula A of the modal language $\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg,\mathbf{I},\mathbf{C}\}}$ the following conditions are equivalent

(i) $\vdash_{S4} A$

(ii) $\models_{S4} A$

(iii) A is valid in every topological field of sets $\mathcal{B}(X)$

(iv) A is valid in every topological Boolean algebra $\mathcal{B}$ with at most 2^{2^r} elements, where r is the number of all sub formulas of A

(iv) $v^*(A) = X$ for every variable assignment v in the topological field of sets $\mathcal{B}(X)$ of all subsets of a dense-in -itself metric space $X \neq \emptyset$ (in particular of an n-dimensional Euclidean space X)

S4 Algebraic Completeness Theorem

S5 Algebraic Completeness Theorem

For any formula A of the modal language $\mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg,\mathbf{I},\mathbf{C}\}}$ the following conditions are equivalent

(i) $\vdash_{S5} A$

(ii) $\models_{S5} A$

(iii) A is valid in every **clopen** topological field of sets $\mathcal{B}(X)$

(iv) A is valid in every **clopen** topological Boolean algebra $\mathcal{B}$ with at most 2^{2^r} elements, where r is the number of all sub formulas of A

S4 and S5 Decidability

The equivalence of conditions **(i)** and **(iv)** of the Algebraic Completeness Theorems proves the **semantical** decidability of modal S4 and S5

S4, S5 Decidability

Any complete S4, S5 proof system is **semantically decidable**, i.e. the following holds

$$\vdash_{S4} A \quad \text{if and only if} \quad \mathcal{B} \models A$$

for every topological Boolean algebra $\mathcal{B}$ with at most 2^{2^r} elements, where r is the number of all sub formulas of A

Similarly, we also have

$$\vdash_{S5} A \quad \text{if and only if} \quad \mathcal{B} \models A$$

for every **clopen** topological Boolean algebra $\mathcal{B}$ with at most 2^{2^r} elements, where r is the number of all sub formulas of A

S4 and S5 Syntactic Decidability

S4, S5 Syntactic Decidability (Wasilewska 1967,1971)

Rasiowa stated in 1950 an **an open problem** of providing a cut-free RS type formalization for modal propositional S4 calculus

Wasilewska solved this open problem in 1967 and presented the result at the ASL Summer School and Colloquium in Mathematical Logic, Manchester, August 1969

It appeared in print as *A Formalization of the Modal Propositional S4-Calculus*, Studia Logica, North Holland, XXVII (1971)

S4 and S5 Syntactic Decidability

The paper also contained an algebraic proof of **completeness** theorem followed by Gentzen cut-elimination theorem, the **Hauptzatz**

The resulting implementation written in LISP-ALGOL was the **first** modal logic **theorem prover** created

It was done with collaboration with B. Waligorski and the authors didn't think it to be worth a separate publication

Its existence was only mentioned in the published paper

The S5 Syntactic Decidability follows from the one for S4 and the following **Embedding Theorems**

Modal S4 and Modal S5

The relationship between S4 and S5 was **first** established by Ohnishi and Matsumoto in 1957-59 and is as follows .

Embedding 1

For any formula $A \in \mathcal{F}$,

$$\models_{S4} A \quad \text{if and only if} \quad \models_{S5} \mathbf{IC}A$$

$$\vdash_{S4} A \quad \text{if and only if} \quad \vdash_{S5} \mathbf{IC}A$$

Embedding 2

For any formula $A \in \mathcal{F}$

$$\models_{S5} A \quad \text{if and only if} \quad \models_{S4} \mathbf{ICI}A$$

$$\vdash_{S5} A \quad \text{if and only if} \quad \vdash_{S4} \mathbf{ICI}A$$

On S4 derivable disjunction

In a classical logic it is possible for the disjunction $(A \cup B)$ to be a tautology when **neither** A **nor** B is a tautology

This does not hold for the intuitionistic logic. We have a following theorem similar to the intuitionistic case to the for modal S4

Theorem McKinsey, Tarski (1948)

A disjunction $(IA \cup IB)$ is S4 **provable** if and only if either A or B S4 **provable**, i.e.

$$\vdash_{S4} (IA \cup IB) \quad \text{if and only if} \quad \vdash_{S4} A \quad \text{or} \quad \vdash_{S4} B$$

S4 and Intuitionistic Logic, S5 and Classical Logic

S4 and Intuitionistic Logic

As we have said in the introduction, Gödel was the first to consider the **connection** between the intuitionistic logic and a logic which was named later S4

Gödel's proof was purely **syntactic** in its nature, as the semantics for neither intuitionistic logic nor modal logicS4 had not been invented yet

The **algebraic** proof of this fact, was first published by McKinsey and Tarski in 1948

S4 and Intuitionistic Logic

We define here the Gödel-Tarski **mapping** establishing the S4 and intuitionistic logic connection

We refer the reader to Rasiowa, Sikorski book ”Mathematics of Metamathematics” (i965) for the algebraic proofs of its properties and respective theorems

S4 and Intuitionistic Logic

Let $\mathcal{L}$ be a propositional language of modal logic i.e the language

$$\mathcal{L} = \mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg,\mathbf{I}\}}$$

Let $\mathcal{L}_0$ be a language obtained from $\mathcal{L}$ by elimination of the connective $\mathbf{I}$ and by the replacement the classical negation connective $\neg$ by the intuitionistic negation, which we will **denote** here by a symbol $\sim$

Such obtained language

$$\mathcal{L}_0 = \mathcal{L}_{\{\cap,\cup,\Rightarrow,\sim\}}$$

is a propositional language of the intuitionistic logic

S4 and Intuitionistic Logic

In order to establish the connection between the languages

$$\mathcal{L} \quad \text{and} \quad \mathcal{L}_0$$

and hence between modal and intuitionistic logic, we consider a **mapping** f which to every formula $A \in \mathcal{F}_0$ of $\mathcal{L}_0$ **assigns** a formula $f(A) \in \mathcal{F}$ of $\mathcal{L}$

We define the **mapping** f as follows

Gödel - Tarski Mapping

Definition of Gödel-Tarski **mapping**

A function

$$f: \mathcal{F}_0 \quad \rightarrow \mathcal{F}$$

such that

$$f(a) = \mathbf{I}a \quad \textit{for any} \quad a \in VAR$$

$$f((A \Rightarrow B)) = \mathbf{I}(f(A) \Rightarrow f(B))$$

$$f((A \cup B)) = (f(A) \cup f(B))$$

$$f((A \cap B)) = (f(A) \cap f(B))$$

$$f(\sim A) = \mathbf{I}\neg f(A)$$

where A, B are any formulas in $\mathcal{L}_0$ is called a Gödel-Tarski **mapping**

Example

Let A be a formula

$$((\sim A \cap \sim B) \Rightarrow \sim (A \cup B))$$

and f be the Gödel-Tarski mapping. We evaluate $f(A)$ as follows

$$f((\sim A \cap \sim B) \Rightarrow \sim (A \cup B)) =$$

$$\mathbf{I}(f(\sim A \cap \sim B) \Rightarrow f(\sim (A \cup B)) =$$

$$\mathbf{I}((f(\sim A) \cap f(\sim B)) \Rightarrow f(\sim (A \cup B)) =$$

$$\mathbf{I}((\mathbf{I}\neg fA \cap \mathbf{I}\neg fB) \Rightarrow \mathbf{I}\neg f(A \cup B)) =$$

$$\mathbf{I}((\mathbf{I}\neg A \cap \mathbf{I}\neg B) \Rightarrow \mathbf{I}\neg(fA \cup fB)) =$$

$$\mathbf{I}((\mathbf{I}\neg A \cap \mathbf{I}\neg B) \Rightarrow \mathbf{I}\neg(A \cup B))$$

S4 and Intuitionistic Logic

The following theorem established relationship between intuitionistic and modal $S4$ logics

Theorem

Let f be the Gödel-Tarski **mapping**

For any formula A of intuitionistic language $\mathcal{L}_0$,

$$\vdash_I A \quad \text{if and only if} \quad \vdash_{S4} f(A)$$

where I, $S4$ denote any proof systems for intuitionistic and and S4 logic, respectively

Classical Logic and Modal *S5*

In order to establish the connection between the modal S5 and classical logics we consider the following G'odel-Tarski **mapping** between the modal language $\mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg,\mathbf{I}\}}$ and its classical sub-language $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$

With every **classical** formula A we associate a **modal** formula $g(A)$ defined by induction on the length of A as follows:

$$g(a) = \mathbf{I}a, \quad g((A \Rightarrow B)) = \mathbf{I}(g(A) \Rightarrow g(B),)$$

$$g((A \cup B)) = (g(A) \cup g(B)), \quad g((A \cap B)) = (g(A) \cap g(B)),$$

$$g(\neg A) = \mathbf{I}\neg g(A)$$

Classical Logic and Modal *S5*

The following theorem establishes **relationship** between classical and *S5* logics

Theorem

Let g be the Gödel-Tarski **mapping** between

$$\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}} \quad \text{and} \quad \mathcal{L}_{\{\cap,\cup,\Rightarrow,\neg,\mathbf{I}\}}$$

For any formula A of $\mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$,

$$\vdash_H A \quad \text{if and only if} \quad \vdash_{S5} g(A)$$

where $H, S5$ denote any proof systems for classical and and S5 modal logic, respectively