CSE371 Extra Q3 SOLUTIONS Spring 2024
(3pts extra credit)

ONE PROBLEM PART 1 (1.5pts)

Let $\mathcal{L} = \mathcal{L}_{\{\neg,\ \sim,\ \Rightarrow,\ \rightarrow\}}$ be a language with one argument connectives $\neg$, $\sim$ called **strong negation** and **weak negation**, and with two arguments connectives $\Rightarrow$, $\rightarrow$ called **strong implication** and **weak implication**.

We define a **3 valued** extensional semantics **M** for the language $\mathcal{L}_{\{\neg,\ \sim,\ \Rightarrow,\ \rightarrow\}}$ by **defining the connectives** $\neg$, $\sim$, $\Rightarrow$, $\rightarrow$ as functions on the set $\{F, \perp, T\}$ of 3 logical values as follows.

The functions $\neg$, $\Rightarrow$ **restricted** to the set $\{F, T\}$ are the same as in the **classical case**.

We extend them to the full set $\{F, \perp, T\}$ for **strong negation** as $\neg \perp = F$, and for **strong implication** as $x \Rightarrow \perp = F$ for $x = T, F$ and

$$\perp \Rightarrow y = \begin{cases} \perp & \text{if } y = \perp \\ T & \text{otherwise} \end{cases}$$

We define the **weak negation** $\sim$: $\{T, \perp, F\} \longrightarrow \{T, \perp, F\}$ as

$$\sim x = \begin{cases} T & \text{if } x = \perp \\ x & \text{for } x \in \{T, F\} \end{cases}$$

The **weak implication** $\rightarrow$: $\{T, \perp, F\} \times \{T, \perp, F\} \longrightarrow \{T, \perp, F\}$ is defined for all $x, y \in \{T, \perp, F\}$ as $x \rightarrow y = \sim (x \Rightarrow y)$

Fill in the connectives tables. Remember that the **M** connectives $\neg$, $\Rightarrow$ on set $\{F, T\}$ are the same as **classical** $\neg$, $\Rightarrow$.

$\neg$	F	$\perp$	T
	T	F	F

$\sim$	F	$\perp$	T
	F	T	T

$\Rightarrow$	F	$\perp$	T
F	T	F	T
$\perp$	T	$\perp$	T
T	F	F	T

$\rightarrow$	F	$\perp$	T
F	T	F	T
$\perp$	T	T	T
T	F	F	T

ONE PROBLEM PART 2 (1.5pts) Use **shorthand notation**.

(0.5pts) Prove that $\not\models_{\mathbf{M}} (a \Rightarrow a)$ and $\models_{\mathbf{M}} (a \Rightarrow \neg\neg a)$.

Solution To prove $\not\models_{\mathbf{M}} (a \Rightarrow a)$ we have to find a counter MODEL v for $(a \Rightarrow \neg\neg a)$.

Consider any $v : VAR \longrightarrow \{F, \perp, T\}$ such that $v(a) = \perp$.

We evaluate $\perp \Rightarrow \perp = F$ and so $(a \Rightarrow a)$ is not a **M** tautology.

To prove that $\models_{\mathbf{M}} (a \Rightarrow \neg\neg a)$ we first observe that it is a classical tautology and the **M** connectives $\neg$, $\Rightarrow$ on set $\{F, T\}$ are the same as **classical** $\neg$, $\Rightarrow$, so to prove $\models_{\mathbf{M}} (a \Rightarrow \neg\neg a)$ we have to consider only the case $a = \perp$ and get $\perp \Rightarrow \neg\neg \perp = \perp \Rightarrow \neg F = \perp \Rightarrow T = T$.

This ends the proof.

(0.5pts) Let **T** be a set of classical tautologies, **LT** be a set of Lukasiewicz tautologies, and **MT** be a set of all **M** tautologies.

Prove that $\mathbf{T} \cap \mathbf{MT} \neq \emptyset$ and $\mathbf{LT} \neq \mathbf{MT}$

Solution We just proved that the formula $(a \Rightarrow \neg\neg a) \in \mathbf{T} \cap \mathbf{MT}$, hence $\mathbf{T} \cap \mathbf{MT} \neq \emptyset$.

As we have proved that $\not\models_{\mathbf{M}} (a \Rightarrow a)$, and we know that $(a \Rightarrow a) \in \mathbf{LT}$ we proved that $\mathbf{LT} \neq \mathbf{MT}$.

(0.5pts) Prove that the semantics **M** is **well defined**

Solution By definition, semantics **M** is **well defined** if and only if $\mathbf{MT} \neq \emptyset$.

This is true as we have already proved that $(a \Rightarrow \neg\neg a) \in \mathbf{MT}$.