CSE371/Math371 Q5 SOLUTIONS Spring 2024
(3pts extra credit)

ONE PROBLEM PART 1 (1pts)

Let S be the following **proof system** $S = (\mathcal{L}_{\{\neg,\ \mathbf{L},\ \cup,\ \Rightarrow\}},\ \mathcal{F},\ \{\mathbf{A1}, \mathbf{A2}\},\ \{r1,\ r2\})$

for the logical axioms and rules of inference defined for **any formulas** $A, B \in \mathcal{F}$ as follows

Logical Axioms **A1** $(\mathbf{L}A \cup \neg \mathbf{L}A)$, **A2** $(A \Rightarrow \mathbf{L}A)$

Rules of inference: $(r1)\ \frac{A\ ;\ B}{(A \cup B)}$ $(r2)\ \frac{A}{\mathbf{L}(A \Rightarrow B)}$

Show, by constructing a proper **formal proof** that $\vdash_S ((\mathbf{L}b \cup \neg \mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg \mathbf{L}a) \Rightarrow b)))$

Write all steps of the **formal proof** with **comments** how each step was obtained.

Solution

Here is the proof $B_1, B_2,\ B_3,\ B_4$

B_1: $(\mathbf{L}a \cup \neg \mathbf{L}a)$ Axiom A_1 for A= a

B_2: $\mathbf{L}((\mathbf{L}a \cup \neg \mathbf{L}a) \Rightarrow b)$ rule r2 for B=b applied to B_1

B_3: $(\mathbf{L}b \cup \neg \mathbf{L}Ab)$ Axiom A_1 for A=b

B_4: $((\mathbf{L}b \cup \neg \mathbf{L}b) \cup \mathbf{L}((\mathbf{L}a \cup \neg \mathbf{L}a) \Rightarrow b))$ r1 applied to B_3 and B_2

ONE PROBLEM PART 2 (2pts)

Consider the Hilbert system $H1 = (\mathcal{L}_{\{\Rightarrow\}},\ \mathcal{F},\ \{A1, A2\},\ (MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B})$ where for any $A, B \in \mathcal{F}$
$A1;\ (A \Rightarrow (B \Rightarrow A)),\quad A2:\ ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$

1. (1pts) Use the **Deduction Theorem** (we know that is holds for $H1$) show that

$$(\neg a \Rightarrow ((b \Rightarrow \neg a) \Rightarrow b)) \vdash_{H1} ((b \Rightarrow \neg a) \Rightarrow (\neg a \Rightarrow b))$$

Hint First apply Deduction Theorem twice

Solution

$(\neg a \Rightarrow ((b \Rightarrow \neg a) \Rightarrow b)) \vdash_{H1} ((b \Rightarrow \neg a) \Rightarrow (\neg a \Rightarrow b))$ if and only if

$(\neg a \Rightarrow ((b \Rightarrow \neg a) \Rightarrow b)),\ (b \Rightarrow \neg a) \vdash_{H1} (\neg a \Rightarrow b))$ if and only if

$(\neg a \Rightarrow ((b \Rightarrow \neg a) \Rightarrow b)),\ (b \Rightarrow \neg a),\ \neg a \vdash_{H1} b$

We now construct a proof of $(\neg a \Rightarrow ((b \Rightarrow \neg a) \Rightarrow b)),\ (b \Rightarrow \neg a),\ \neg a \vdash_{H1} b$ as follows

B_1 : $(\neg a \Rightarrow ((b \Rightarrow \neg a) \Rightarrow b))$ hypothesis

B_2 : $(b \Rightarrow \neg a)$ hypothesis

B_3 : $\neg a$ hypothesis

B_4 : $((b \Rightarrow \neg a) \Rightarrow b))$ B_1 , B_3 and (MP)

B_5 : b B_2, B_4 and (MP)

2. (1pts) Let $H2$ be a **complete the proof system** obtained from the system $H1$ by **extending the language** to contain the negation $\neg$ and **adding** one additional axiom:

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

Let $H3$ be the proof system obtained from the system $H2$ **adding** additional axiom

A4 $(\neg(A \Rightarrow B) \Rightarrow \neg(A \Rightarrow \neg B))$

Does **Completeness Theorem** hold for $H3$? JUSTIFY.

Solution

No, **it doesn't**. The system H3 is **not sound**

Axiom **A4** is not a tautology.

Any v such that A=T and B=F is a **counter model** for $(\neg(A \Rightarrow B) \Rightarrow \neg(A \Rightarrow \neg B))$.

$(\neg(T \Rightarrow F) \Rightarrow \neg(T \Rightarrow \neg F)) = \neg F \Rightarrow \neg T = T \Rightarrow F = F$.