

Chapter 4: Classical Propositional Semantics

Language :

$$\mathcal{L}\{\neg, \cup, \cap, \Rightarrow\}.$$

Classical Semantics assumptions:

TWO VALUES: there are only two logical values: truth (T) and false (F), and

EXTENSIONALITY: the logical value of a formula depends only on a main connective and logical values of its sub-formulas.

We define formally a classical semantics for \mathcal{L} in terms of two factors: classical truth tables and a truth assignment.

We summarize now here the chapter 2 tables for $\mathcal{L}\{\neg, \cup, \cap, \Rightarrow\}$ in one simplified table as follows.

A	B	$\neg A$	$(A \cap B)$	$(A \cup B)$	$(A \Rightarrow B)$
T	T	F	T	T	T
T	F	F	F	T	F
F	T	T	F	T	T
F	F	T	F	F	T

Observe that The first row of the above table reads:

For any formulas A, B , if the logical value of $A = T$ and $B = T$, then logical values of $\neg A = F$, $(A \cap B) = T$, $(A \cup B) = T$ and $(A \Rightarrow B) = T$.

We read and write the other rows in a similar manner.

Our table indicates that the logical value of of propositional connectives depends **only** on the logical values of its factors; i.e. it is **independent of the formulas** A, B .

EXTENSIONAL CONNECTIVES : The logical value of a given connective depend only of the logical values of its factors.

We write now the last table as the following equations.

$$\neg T = F, \quad \neg F = T;$$

$$(T \cap T) = T, \quad (T \cap F) = F, \quad (F \cap T) = F, \quad (F \cap F) = F;$$

$$(T \cup T) = T, \quad (T \cup F) = T, \quad (F \cup T) = T, \quad (F \cup F) = F;$$

$$(T \Rightarrow T) = T, \quad (T \Rightarrow F) = F, \quad (F \Rightarrow T) = T, \quad (F \Rightarrow F) = T.$$

Observe now that the above equations describe a set of unary and binary operations (functions) defined on a set $\{T, F\}$ and a set $\{T, F\} \times \{T, F\}$, respectively.

Negation \neg is a function:

$$\neg : \{T, F\} \longrightarrow \{T, F\},$$

such that $\neg T = F$, $\neg F = T$.

Conjunction \cap is a function:

$$\cap : \{T, F\} \times \{T, F\} \longrightarrow \{T, F\},$$

such that

$$(T \cap T) = T, \quad (T \cap F) = F,$$

$$(F \cap T) = F, \quad (F \cap F) = F.$$

Dissjunction \cup is a function:

$$\cup: \{T, F\} \times \{T, F\} \longrightarrow \{T, F\},$$

such that

$$\begin{aligned}(T \cup T) &= T, & (T \cup F) &= T, \\ (F \cup T) &= T, & (F \cup F) &= F.\end{aligned}$$

Implication \Rightarrow is a function:

$$\Rightarrow: \{T, F\} \times \{T, F\} \longrightarrow \{T, F\},$$

such that

$$\begin{aligned}(T \Rightarrow T) &= T, & (T \Rightarrow F) &= F, \\ (F \Rightarrow T) &= T, & (F \Rightarrow F) &= T.\end{aligned}$$

Observe that if we have a language

$\mathcal{L}_{\{\neg, \cup, \cap, \Rightarrow, \Leftrightarrow\}}$ containing also the equivalence
connective \Leftrightarrow we define

$$\Leftrightarrow: \{T, F\} \times \{T, F\} \longrightarrow \{T, F\},$$

as a function such that

$$\begin{aligned}(T \Leftrightarrow T) &= T, & (T \Leftrightarrow F) &= F, \\ (F \Leftrightarrow T) &= F, & (F \Leftrightarrow F) &= T.\end{aligned}$$

We write these definitions of connectives as the following tables, usually called the **classical truth tables**.

Negation : **Disjunction :**

\neg	T	F
	F	T

\cup	T	F
T	T	T
F	T	F

Conjunction : **Implication :**

\cap	T	F
T	T	F
F	F	F

\Rightarrow	T	F
T	T	F
F	T	T

Equivalence :

\Leftrightarrow	T	F
T	T	F
F	F	T

A truth assignment is any function

$$v : VAR \longrightarrow \{T, F\}.$$

Observe that the truth assignment is defined only on variables (atomic formulas).

We define its **extension** v^* to the set \mathcal{F} of all formulas of \mathcal{L} as follows.

$$v^* : \mathcal{F} \longrightarrow \{T, F\}$$

is such that

(i) for any $a \in VAR$,

$$v^*(a) = v(a);$$

(ii) and for any $A, B \in \mathcal{F}$,

$$v^*(\neg A) = \neg v^*(A);$$

$$v^*(A \cap B) = v^*(A) \cap v^*(B);$$

$$v^*(A \cup B) = v^*(A) \cup v^*(B);$$

$$v^*(A \Rightarrow B) = v^*(A) \Rightarrow v^*(B),$$

$$v^*(A \Leftrightarrow B) = v^*(A) \Leftrightarrow v^*(B),$$

where

the symbols on the **left-hand side** of the equations represent connectives in their **natural language meaning** and

the symbols on the **right-hand side** represent connectives in their **logical meaning** given by the classical truth tables.

Example

Consider a formula

$$((a \Rightarrow b) \cup \neg a))$$

a truth assignment v such that

$$v(a) = T, v(b) = F.$$

We calculate the logical value of the formula

$$\begin{aligned} A \text{ as follows: } v^*(A) &= v^*((a \Rightarrow b) \cup \neg a) = \\ &v^*(a \Rightarrow b) \cup v^*(\neg a) = (v(a) \Rightarrow v(b)) \cup \neg v(a) = \\ &(T \Rightarrow F) \cup \neg T = F \cup F = F. \end{aligned}$$

Observe that we did not need (and usually we don't) to specify the $v(x)$ of any $x \in VAR - \{a, b\}$, as these values do not influence the computation of the logical value $v^*(A)$.

SATISFACTION relation

Definition: Let $v : VAR \longrightarrow \{T, F\}$. We say that

v **satisfies a formula** $A \in \mathcal{F}$ iff $v^*(A) = T$

Notation: $v \models A$.

Definition: We say that

v **does not satisfy a formula** $A \in \mathcal{F}$ iff $v^*(A) \neq T$.

Notation: $v \not\models A$.

REMARK In our classical semantics we have that

$v \not\models A$ iff $v^*(A) = F$ and we say that v **falsifies the formula** A .

OBSERVE $v^*(A) \neq T$ is equivalent to the fact that $v^*(A) = F$ ONLY in 2-valued logic!

This is why we adopt the following

Definition: For any v ,
 v does not satisfy a formula $A \in \mathcal{F}$ iff
 $v^*(A) \neq T$

Example

$$A = ((a \Rightarrow b) \cup \neg a))$$

$$v : VAR \longrightarrow \{T, F\}$$

such that $v(a) = T, v(b) = F$.

Calculation of $v^*(A)$ using the short hand notation:

$$(T \Rightarrow F) \cup \neg T = F \cup F = F.$$

$$v \not\models ((a \Rightarrow b) \cup \neg a).$$

Observe that we did not need (and usually we don't) to specify the $v(x)$ of any $x \in VAR - \{a, b\}$, as these values do not influence the computation of the logical value $v^*(A)$.

Example

$$A = ((a \wedge \neg b) \vee \neg c)$$

$$v : VAR \longrightarrow \{T, F\}$$

such that $v(a) = T, v(b) = F, v(c) = T$.

Calculation in a short hand notation:

$$(T \wedge \neg F) \vee \neg T = (T \wedge T) \vee F = T \vee F = T.$$

$$v \models ((a \wedge \neg b) \vee \neg c).$$

Formula: $A = ((a \wedge \neg b) \vee \neg c)$.

Consider now $v_1 : VAR \longrightarrow \{T, F\}$ such that
 $v_1(a) = T, v_1(b) = F, v_1(c) = T$, and
 $v_1(x) = F$, for all $x \in VAR - \{a, b, c\}$,

Observe: $v(a) = v_1(a), v(b) = v_1(b), v(c) = v_1(c)$, so we get

$$v_1 \models ((a \wedge \neg b) \vee \neg c).$$

Consider $v_2 : VAR \longrightarrow \{T, F\}$ such that
 $v_2(a) = T, v_2(b) = F, v_2(c) = T, v_2(d) = T,$
and
 $v_2(x) = F,$ for all $x \in VAR - \{a, b, c, d\},$

Observe: $v(a) = v_2(a), v(b) = v_2(b), v(c) =$
 $v_2(c),$ so we get

$$v_2 \models ((a \wedge \neg b) \vee \neg c).$$

We are going to prove that there are as many of such truth assignments as real numbers! but they are all *the same* as the first v with respect to the formula A .

When we ask a question: "*How many truth assignments satisfy/fasify a formula A ?*" we mean to find all assignment that are *different on the formula A* , not just different on a set VAR of all variables, as all of our v_1, v_2 's were.

To address and to answer this question formally we first introduce some notations and definitions.

Notation: for any formula A , we denote by

$$VAR_A$$

a set of **all variables that appear in A** .

Definition: Given a formula $A \in \mathcal{F}$, any function

$$w : VAR_A \longrightarrow \{T, F\}$$

is called a **truth assignment restricted to A** .

Example

$$A = ((a \cap \neg b) \cup \neg c)$$

$$VAR_A = \{a, b, c\}$$

Truth assignment restricted to A is any function:

$$w : \{a, b, c\} \longrightarrow \{T, F\}.$$

We use the following theorem to count all possible truth assignment restricted to A .

Counting Functions Theorem (1) For any finite sets A and B , if A has n elements and B has m elements, then there are m^n possible functions that map A into B .

There are $2^3 = 8$ truth assignment restricted to $A = ((a \Rightarrow \neg b) \cup \neg c)$.

General case For any A there are

$$2^{|VAR_A|}$$

possible truth assignments w restricted to A .

All w restricted to A are listed in the table below.

$$A = ((a \wedge \neg b) \vee \neg c)$$

w	a	b	c	$w^*(A)$ computation	$w^*(A)$
w_1	T	T	T	$(T \Rightarrow T) \vee \neg T = T \vee F = T$	T
w_2	T	T	F	$(T \Rightarrow T) \vee \neg F = T \vee T = T$	T
w_3	T	F	F	$(T \Rightarrow F) \vee \neg F = F \vee T = T$	T
w_4	F	F	T	$(F \Rightarrow F) \vee \neg T = T \vee F = T$	T
w_5	F	T	T	$(F \Rightarrow T) \vee \neg T = T \vee F = T$	T
w_6	F	T	F	$(F \Rightarrow T) \vee \neg F = T \vee T = T$	T
w_7	T	F	T	$(T \Rightarrow F) \vee \neg T = F \vee F = F$	F
w_8	F	F	F	$(F \Rightarrow F) \vee \neg F = T \vee T = T$	T

Model for A is a v such that

$$v \models A.$$

$w_1, w_2, w_3, w_4, w_5, w_6, w_8$ are **models** for A .

Counter- Model for A is a v such that

$$v \not\models A.$$

w_7 is a **counter- model** for A .

Tautology :

A is a **tautology** iff any v is a **model** for A , i.e.

$$\forall v (v \models A).$$

Not a tautology :

A is **not a tautology** iff there is $v : VAR \longrightarrow \{T, F\}$, such that v is a **counter-model** for A , i.e.

$$\exists v (v \not\models A).$$

Tautology Notation $\models A$

Example

$$\not\models ((a \wedge \neg b) \vee \neg c)$$

because the truth assignment w_7 is a counter-model for A .

Tautology Verification

Truth Table Method: list and evaluate all possible truth assignments restricted to A .

Example: $(a \Rightarrow (a \cup b))$.

v	a	b	$v^*(A)$ computation	$v^*(A)$
v_1	T	T	$(T \Rightarrow (T \cup T)) = (T \Rightarrow T) = T$	T
v_2	T	F	$(T \Rightarrow (T \cup F)) = (T \Rightarrow T) = T$	T
v_3	F	T	$(F \Rightarrow (F \cup T)) = (F \Rightarrow T) = T$	T
v_4	F	F	$(F \Rightarrow (F \cup F)) = (F \Rightarrow F) = T$	T

for all $v : VAR \longrightarrow \{T, F\}$, $v \models A$, i.e.

$$\models (a \Rightarrow (a \cup b)).$$

Proof by Contradiction Method

One works backwards, trying to find a truth assignment v which makes a formula A false.

If we find one, it means that A is not a tautology,

if we prove that it is impossible ,

it means that the formula is a tautology.

Example $A = (a \Rightarrow (a \cup b))$

Step 1 Assume that $\not\models A$, i.e. $A = F$.

Step 2 Analyze Step 1:

$$(a \Rightarrow (a \cup b)) = F \quad \text{iff} \quad a = T \quad \text{and} \\ a \cup b = F.$$

Step 3 Analyze Step 2:

$$a = T \quad \text{and} \quad a \cup b = F, \text{ i.e. } T \cup b = F.$$

This is impossible by the definition of \cup .

Conclusion:

$$\models (a \Rightarrow (a \cup b)).$$

Observe that exactly the same reasoning proves that for any formulas $A, B \in \mathcal{F}$,

$$\models (A \Rightarrow (A \cup B)).$$

Observe that the following formulas are tautologies

$$(((a \Rightarrow b) \wedge \neg c) \Rightarrow (((a \Rightarrow b) \wedge \neg c) \cup \neg d)),$$

$$(((a \Rightarrow b) \wedge \neg C) \cup d) \wedge \neg e) \Rightarrow$$

$$(((a \Rightarrow b) \wedge \neg C) \cup d) \wedge \neg e) \cup ((a \Rightarrow \neg e)))$$

because they are of the form

$$(A \Rightarrow (A \cup B)).$$

Tautologies, Contradictions

$$\mathbf{T} = \{A \in \mathcal{F} : \models A\},$$

$$\mathbf{C} = \{A \in \mathcal{F} : \forall v (v \not\models A)\}.$$

Theorem 1 For any formula $A \in \mathcal{F}$ the following conditions are equivalent.

(1) A is a tautology

(2) $A \in \mathbf{T}$

(3) $\neg A$ is a contradiction

(4) $\neg A \in \mathbf{C}$

(5) $\forall v (v^*(A) = T)$

(6) $\forall v (v \models A)$

(7) Every v is a model for A

Theorem 2 For any formula $A \in \mathcal{F}$ the following conditions are equivalent.

(1) A is a contradiction

(2) $A \in \mathbf{C}$

(3) $\neg A$ is a tautology

(4) $\neg A \in \mathbf{T}$

(5) $\forall v (v^*(A) = F)$

(6) $\forall v (v \not\models A)$

(7) A does not have a model.