CHAPTER 8

System H_2 and Formal Proofs

Hilbert System H_2

The system H_1 is sound and strong enough to prove the Deduction Theorem, but it is not complete.

We extend now its set of logical axioms to **a complete set of axioms**, i.e. we define a system H_2 that is **complete** with respect to classical semantics.

The proof of completeness will be presented in the next chapter.

Definition of the system H_2.

$$H_2 = (\ \mathcal{L}_{\{\Rightarrow, \neg\}}, \quad A1, A2, A3, \quad MP\ )$$

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B},$$

A, B, C are any formulas of the propositional language $\mathcal{L}_{\{\Rightarrow, \neg\}}$.

We write

$$\vdash_{H_2} A$$

to denote that a formula A has a formal proof in H_2 (from the set of logical axioms $A1, A2, A3$), and

$$\Gamma \vdash_{H_2} A$$

to denote that a formula A has a formal proof in H_2 from a set of formulas Γ (and the set of logical axioms $A1, A2, A3$).

Observe that system H_2 is obtained by adding axiom A_3 to the system H_1.

Hence the Deduction Theorem holds for system H_2.

Deduction Theorem for H_2

For any subset Γ of the set of formulas $\mathcal{F}$ of H_2 and for any formulas $A, B \in \mathcal{F}$,

Γ, $A \vdash_{H_2} B$ if and only if $\Gamma \vdash_{H_2} (A \Rightarrow B)$.

In particular,

$$A \vdash_{H_2} B \ \ if\ and\ only\ if\ \ \vdash_{H_2} (A \Rightarrow B).$$

Obviously, the axioms $A1, A2, A3$ are tautologies, and the Modus Ponens rule leads from tautologies to tautologies, hence our proof system H_2 is *sound* i.e. the following theorem holds.

Soundness Theorem for H_2

For every formula $A \in \mathcal{F}$,

$$\text{if } \vdash_{H_2} A, \text{ then } \models A.$$

The soundness theorem proves that the system ''produces'' only tautologies. We show, in the next chapter, that our proof system H_2 ''produces'' not only tautologies, but that all tautologies are provable in it. This is called a **completeness theorem** for classical logic.

Completeness Theorem for H_2

For every $A \in \mathcal{F}$,

$$\vdash_{H_2} A, \quad \text{if and only if} \quad \models A.$$

The proof of completeness theorem (for a given semantics) is always a main point in any logic creation.

There are many ways (techniques) to prove it, depending on the proof system, and on the semantics we define for it.

We present in the next chapter two proofs of the completeness theorem for our system H_2.

The proofs use very different techniques, hence the reason of presenting both of them.

In fact the proofs are valid for any proof system for classical propositional logic in which one can prove all formulas proved in the next section.

FORMAL PROOFS IN H_2
Examples and Exercises

We present here some examples of formal proofs in H_2. There are two reasons for presenting them.

First reason is that all formulas we prove here to be provable play a crucial role in the proof of Completeness Theorem for H_2, or are needed to find formal proofs of those needed.

The second reason is that they provide a "training" ground for a reader to learn how to develop formal proofs.

For this reason we write some proofs in a full detail and we leave some for the reader to complete in a way explained in the following example.

We write $\vdash$ instead of $\vdash_{H_2}$ for the sake of simplicity.

Reminder In the construction of the formal proofs we very often use Deduction Theorem and the following Lemma (proved in previous section)

Lemma 1 :

$$\textbf{(a)} \quad (A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C),$$

$$\textbf{(b)} \quad (A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} ((B \Rightarrow (A \Rightarrow C)).$$

EXAMPLE 1

Here are consecutive steps

$B_1, ..., B_5, B_6$

of the proof (in H_2) of $(\neg\neg B \Rightarrow B)$.

$$B_1 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$$

$$B_2 = ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$$

$$B_3 = (\neg B \Rightarrow \neg B)$$

$$B_4 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$$

$$B_5 = (\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$$

$$B_6 = (\neg\neg B \Rightarrow B).$$

EXERCISE 1

Complete the proof presented in the example 1 by providing comments how each step of the proof was obtained.

ATTENTION The solution presented here shows you how you will have to write details of YOUR solutions on the TESTS.

Solutions of other problems presented later are less detailed. Use them as exercises to write a detailed, complete solution.

Solution

The comments that complete the proof are as follows.

$B_1 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$
Axiom $A3$ for $A = \neg B, B = B$

$B_2 = ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

B_1 and lemma 1 **b** for $A = (\neg B \Rightarrow \neg\neg B), B = (\neg B \Rightarrow \neg B), C = B$, i.e.

$((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B)) \vdash ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

$B_3 = (\neg B \Rightarrow \neg B)$
We proved for H_1 and hence for H_2 that $\vdash (A \Rightarrow A)$ and we substitute $A = \neg B$

$B_4 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$
B_2, B_3 and MP

$B_5 = (\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$
Axiom $A1$ for $A = \neg\neg B, B = \neg B$

$B_6 = (\neg\neg B \Rightarrow B)$

B_4, B_5 and Lemma 1 **a** for $A = \neg\neg B, B = (\neg B \Rightarrow \neg\neg B), C = B$; i.e.

$(\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B)),\ ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)\ \vdash\ (\neg\neg B \Rightarrow B)$.

GENERAL REMARK

In step B_2, B_3, B_5, B_6 we call previously proved facts and use their results as a part of our proof. We can insert previously constructed formal proofs into our formal proof.

For example we adopt previously constructed proof of $(A \Rightarrow A)$ in H_1 to the proof of $(\neg B \Rightarrow \neg B)$ in H_2 by replacing A by $\neg B$ and we insert the proof of $(\neg B \Rightarrow \neg B)$ after B_2.

The "old" step B_3 becomes now B_7, the "old" step B_4 becomes now B_8, etc.....

$B_1 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$
Axiom $A3$ for $A = \neg B, B = B$

$B_2 = ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

B_1 and lemma 1 **b** for $A = (\neg B \Rightarrow \neg\neg B), B = (\neg B \Rightarrow \neg B), C = B$,

$B_3 = ((\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B)) \Rightarrow ((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B)))$,
axiom A2 for $A = \neg B$, $B = (\neg B \Rightarrow \neg B)$, and $C = \neg B$

$B_4 = (\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B))$,
axiom A1 for $A = \neg B$, $B = (\neg B \Rightarrow \neg B)$

$B_5 = ((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B)))$,
MP application to B_4 and B_3

$B_6 = (\neg B \Rightarrow (\neg B \Rightarrow \neg B))$,
axiom A1 for $A = \neg B$, $B = \neg B$

$B_7 =$ ("old" B_3)$(\neg B \Rightarrow \neg B)$
MP application to B_5 and B_4

$B_8 =$ ("old" B_4) $((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$
B_2, B_3 and MP

$B_9 =$ ("old B_5) $(\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$
Axiom $A1$ for $A = \neg\neg B, B = \neg B$

$B_{10} =$ ("old B_6) $(\neg\neg B \Rightarrow B)$
B_8, B_9 and Lemma 1 **a** for $A = \neg\neg B, B = (\neg B \Rightarrow \neg\neg B), C = B$

We repeat our procedure by replacing the step B_2 by its formal proof as defined in the proof of the lemma 1 **b**, and continue the process for all other steps which involved application of lemma 1 until we get a full formal proof from the axioms of H_2 only.

Usually we don't need to do it, but it is important to remember that it always can be done, if we wished to take time and space to do so.

EXAMPLE 2

Here are consecutive steps

$B_1, ..., B_5$ in a proof of $(B \Rightarrow \neg\neg B)$.

$$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$$

$$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$$

$$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$$

$$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$$

$$B_5 = (B \Rightarrow \neg\neg B)$$

EXERCISE 2

Complete the proof presented in Example 2 by providing detailed comments how each step of the proof was obtained.

Solution

The comments that complete the proof are as follows.

$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$
Axiom $A3$ for $A = B, B = \neg\neg B$

$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$
Example 1 for $B = \neg B$

$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$

B_1, B_2 and MP, i.e.

$$\frac{(\neg\neg\neg B \Rightarrow \neg B); ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))}{((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)}$$

$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$

Axiom $A1$ for $A = B, B = \neg\neg\neg B$

$B_5 = (B \Rightarrow \neg\neg B)$

B_3, B_4 and lemma 1**a** for $A = B, B = (\neg\neg\neg B \Rightarrow B), C = \neg\neg B$, i.e.

$(B \Rightarrow (\neg\neg\neg B \Rightarrow B)), ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B) \vdash_{H_2} (B \Rightarrow \neg\neg B)$

EXAMPLE 3

Here are consecutive steps $B_1, ..., B_{12}$ in a proof of $(\neg A \Rightarrow (A \Rightarrow B))$.

$B_1 = \neg A$

$B_2 = A$

$B_3 = (A \Rightarrow (\neg B \Rightarrow A))$

$B_4 = (\neg A \Rightarrow (\neg B \Rightarrow \neg A))$

$B_5 = (\neg B \Rightarrow A)$

$B_6 = (\neg B \Rightarrow \neg A)$

$B_7 = ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

$B_8 = ((\neg B \Rightarrow A) \Rightarrow B)$

$B_9 = B$

$B_{10} = \neg A, A \vdash B$

$B_{11} = \neg A \vdash (A \Rightarrow B)$

$B_{12} = (\neg A \Rightarrow (A \Rightarrow B))$

EXERCISE 3

1. Complete the proof from the example 3 by providing comments how each step of the proof was obtained.

2. Prove that $\neg A, A \vdash B$.

EXAMPLE 4

Here are consecutive steps $B_1, ..., B_7$ in a proof of $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$.

$B_1 = (\neg B \Rightarrow \neg A)$

$B_2 = ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

$B_3 = (A \Rightarrow (\neg B \Rightarrow A))$

$B_4 = ((\neg B \Rightarrow A) \Rightarrow B)$

$B_5 = (A \Rightarrow B)$

$B_6 = (\neg B \Rightarrow \neg A) \vdash (A \Rightarrow B)$

$B_7 = ((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$

Exercise 4

Complete the proof from example 4 by providing comments how each step of the proof was obtained.

EXAMPLE 5

Here are consecutive steps $B_1, ..., B_9$ in a proof of $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$.

$B_1 = (A \Rightarrow B)$

$B_2 = (\neg\neg A \Rightarrow A)$

$B_3 = (\neg\neg A \Rightarrow B)$

$$B_4 = (B \Rightarrow \neg\neg B)$$

$$B_5 = (\neg\neg A \Rightarrow \neg\neg B)$$

$$B_6 = ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$$

$$B_7 = (\neg B \Rightarrow \neg A)$$

$$B_8 = (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$$

$$B_9 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$$

EXERCISE 5

Complete the proof of example 5 by providing comments how each step of the proof was obtained.

Solution

$B_1 = (A \Rightarrow B)$
Hypothesis

$B_2 = (\neg\neg A \Rightarrow A)$
Example 1 for $B = A$

$B_3 = (\neg\neg A \Rightarrow B)$
Lemma 1 **a** for $A = \neg\neg A, B = A, C = B$

$B_4 = (B \Rightarrow \neg\neg B)$
Example 2

$B_5 = (\neg\neg A \Rightarrow \neg\neg B)$
Lemma 1 **a** for $A = \neg\neg A, B = B, C = \neg\neg B$

$B_6 = ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$
Example 4 for $B = \neg A, A = \neg B$

$B_7 = (\neg B \Rightarrow \neg A)$
B_5, B_6 and MP

$B_8 = (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$
$B_1 - B_7$

$B_9 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
Deduction Theorem

EXERCISE 6

Prove that $\vdash (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B))))$.

Solution Here are consecutive steps of building the formal proof.

$B_1 = A, (A \Rightarrow B) \vdash B$
by MP

$B_2 = A \vdash ((A \Rightarrow B) \Rightarrow B)$
Deduction Theorem

$B_3 = \vdash (A \Rightarrow ((A \Rightarrow B) \Rightarrow B))$
Deduction Theorem

$B_4 = \vdash (((A \Rightarrow B) \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$
Example 5 for $A = (A \Rightarrow B), B = B$

$B_5 = \vdash (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))$

3. and 4. and lemma 2**a** for $A = A, B = ((A \Rightarrow B) \Rightarrow B), C = (\neg B \Rightarrow (\neg(A \Rightarrow B))$

EXAMPLE 7

Here are consecutive steps $B_1, ..., B_{12}$ in a proof of $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$.

$B_1 = (A \Rightarrow B)$

$B_2 = (\neg A \Rightarrow B)$

$B_3 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

$B_4 = (\neg B \Rightarrow \neg A)$

$$B_5 = ((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))$$

$$B_6 = (\neg B \Rightarrow \neg\neg A)$$

$$B_7 = ((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)))$$

$$B_8 = ((\neg B \Rightarrow \neg A) \Rightarrow B)$$

$$B_9 = B$$

$$B_{10} = (A \Rightarrow B), (\neg A \Rightarrow B) \vdash B$$

$$B_{11} = (A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B)$$

$$B_{12} = ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$$

EXERCISE 7

Complete the proof in example 7 by providing comments how each step of the proof was obtained.

Solution

$B_1 = (A \Rightarrow B)$
Hypothesis

$B_2 = (\neg A \Rightarrow B)$
Hypothesis

$B_3 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
Example 5

$B_4 = (\neg B \Rightarrow \neg A)$
B_1, B_3 and MP

$B_5 = ((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))$
Example 5 for $A = \neg A, B = B$

$B_6 = (\neg B \Rightarrow \neg\neg A)$
$B2, B_5$ and MP

$B_7 = ((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)))$
Axiom A3 for $B = B, A = \neg A$

$B_8 = ((\neg B \Rightarrow \neg A) \Rightarrow B)$
B_6, B_7 and MP

$B_9 = B$
B_4, B_8 and MP

$B_{10} = (A \Rightarrow B), (\neg A \Rightarrow B) \vdash B$
$B1 - B9$

$B_{11} = (A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B)$
Deduction Theorem

$B_{12} = ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$ Deduction Theorem

EXAMPLE 8

Here are consecutive steps $B_1, ..., B_3$ in a proof of $((\neg A \Rightarrow A) \Rightarrow A)$.

$$B_1 = ((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A)))$$

$$B_2 = (\neg A \Rightarrow \neg A)$$

$$B_3 = ((\neg A \Rightarrow A) \Rightarrow A))$$

EXERCISE 8

Complete the proof of example 8 by providing comments how each step of the proof was obtained.

Solution

$B_1 = ((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A)))$
Axiom A3 for $B = A$

$B_2 = (\neg A \Rightarrow \neg A)$
Proved $(A \Rightarrow A)$ for $A = \neg A$

$B_3 = ((\neg A \Rightarrow A) \Rightarrow A))$
B_1, B_2 and MP

Examples 1 - 8, and the example 1 of previous section provide a proof of the following lemma.

LEMMA 2 For any formulas A, B, C of the system H_2,

1. $\vdash_{H_2} (A \Rightarrow A)$

2. $\vdash_{H_2} (\neg\neg B \Rightarrow B)$

3. $\vdash_{H_2} (B \Rightarrow \neg\neg B)$

4. $\vdash_{H_2} (\neg A \Rightarrow (A \Rightarrow B))$

5. $\vdash_{H_2} ((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$

6. $\vdash_{H_2} ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

7. $\vdash_{H_2} (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))$

8. $\vdash_{H_2} ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$

9. $\vdash_{H_2} ((\neg A \Rightarrow A) \Rightarrow A$

The set of provable formulas from the above lemma 2 includes a set of provable formulas (formulas 1, 3, 4, and 7-9) needed, with H_2 axioms to execute two proofs of the Completeness Theorem for H_2.

We present these proofs in the next chapter. They represent two diametrally different methods of proving Completeness Theorem.