

# Chapter 9: Completeness

## Theorem: Proof 1

**We consider** a sound proof system (under classical semantics)

$$S = ( \mathcal{L}_{\{\Rightarrow, \neg\}}, \mathcal{AL}, MP ),$$

such that the formulas listed below are provable in  $S$ .

$$\vdash_S (A \Rightarrow (B \Rightarrow A)),$$

$$\vdash_S ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))),$$

$$\vdash_S (\neg A \Rightarrow (A \Rightarrow B)),$$

$$\vdash_S ((\neg A \Rightarrow A) \Rightarrow A),$$

$$\vdash_S ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)),$$

$$\vdash_S (A \Rightarrow A),$$

$$\vdash_S (B \Rightarrow \neg\neg B),$$

$$\vdash_S (A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B))),$$

$$\vdash_S ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)),$$

**We present** here two proofs of the following theorem.

**Completeness Theorem** For any formula  $A$  of  $S$ ,

$$\models A \text{ if and only if } \vdash_S A.$$

**OBSERVATION 1** All the above formulas have proofs in the system  $H_2$  and the system  $H_2$  is sound, hence the Completeness Theorem for the system  $S$  implies the completeness of the system  $H_2$ .

**OBSERVATION 2** We have assumed that the system  $S$  is sound, i.e. that the following theorem holds for  $S$ .

### Soundness Theorem

For any formula  $A$  of  $S$ ,

*if*  $\vdash_S A$ , *then*  $\models A$ .

**It means that** in order to prove the Completeness Theorem we need to prove only the following implication.

**For any formula  $A$  of  $S$ ,**

*If  $\models A$ , then  $\vdash_S A$ .*

**Both proofs** of the Completeness Theorem rely strongly on the Deduction Theorem, as discussed and proved in the previous chapter.

**Deduction theorem** was proved for the system  $H_1$  that is different than  $S$ , but all formulas that were used in its proof are provable in  $S$ , so it is valid for  $S$  as well, as it was for the system  $H_2$ , i.e. the following theorem holds.

**It is possible** to prove the Completeness Theorem independently from the Deduction Theorem and we will present two of such a proof in later chapters.

**The first proof** presented here is similar in its structure to the proof of the deduction theorem and is due to **Kalmar, 1935**.

**It shows** how one can use the assumption that a formula  $A$  is a tautology in order to construct its formal proof. It is hence called a **proof - construction method**.

**The second proof** is a proof of the equivalent opposite implication to the Completeness part, i.e. we show how one can deduce that a formula  $A$  is not a tautology from the fact that it does not have a proof. It is hence called **a counter-model construction method**.

# Completeness Theorem

## A Proof - Construction Method

**We first** present one definition and to prove one lemma.

We write  $\vdash A$  instead of  $\vdash_S A$ , as the system  $S$  is fixed.

**Definition** Let  $A$  be a formula and  $b_1, b_2, \dots, b_n$  be all propositional variables that occur in  $A$ .

Let  $v$  be variable assignment  $v : VAR \longrightarrow \{T, F\}$ .

**We define,** for  $A, b_1, b_2, \dots, b_n$  and  $v$  a corresponding formulas  $A', B_1, B_2, \dots, B_n$  as follows:

$$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases}$$

$$B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$$

for  $i = 1, 2, \dots, n$ .

**Example 1:** let  $A$  be a formula

$$(a \Rightarrow \neg b)$$

**Let**  $v$  be such that

$$v(a) = T, \quad v(b) = F.$$

†

**In this case:**  $b_1 = a$ ,  $b_2 = b$ , and  $v^*(A) = v^*(a \Rightarrow \neg b) = v(a) \Rightarrow \neg v(b) = T \Rightarrow \neg F = T$ .

**The corresponding**  $A', B_1, B_2$  are:

$$A' = A \quad (\text{as } v^*(A) = T),$$

$$B_1 = a \quad (\text{as } v(a) = T),$$

$$B_2 = \neg b \quad (\text{as } v(b) = F).$$

**The lemma stated below** describes a method of transforming a semantic notion of a tautology into a syntactic notion of provability. It defines, for any formula  $A$  and a variable assignment  $v$  a corresponding deducibility relation.

**LEMMA** For any formula  $A$  and a variable assignment  $v$ , if  $A'$ ,  $B_1$ ,  $B_2$ , ...,  $B_n$  are corresponding formulas defined by our definition, then

$$B_1, B_2, \dots, B_n \vdash A'.$$

**For example**, if  $A$ ,  $v$  are as defined by previous example 1, then the lemma asserts that

$$a, \neg b \vdash (a \Rightarrow \neg b).$$

**Example 2** Let  $A, v$  be as defined by the previous example 1, then the Lemma asserts that

$$a, \neg b \vdash (a \Rightarrow \neg b).$$

**Example 3** Let  $A, v$  be as defined in example 1, then the lemma asserts that

$$a, \neg b, \neg c \vdash \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$$

**Proof of the LEMMA** The proof is by induction on the structure of  $A$  i.e. a number  $n$  of logical connectives in  $A$ .

**Case:**  $n = 0$

In the case that  $n = 0$   $A$  is atomic and so consists of a single propositional variable, say  $a$ .

**Clearly,** if  $v^*(A) = T$  then we  $A' = A = a$ ,  
 $B_1 = a$ .

**We obtain** that

$$a \vdash a$$

by the Deduction Theorem and the fact that  $\vdash (A \Rightarrow A)$ , i.e. also  $\vdash (a \Rightarrow a)$ .

**In case** when  $v^*(A) = F$  we have that

$$A' = \neg A = \neg a,$$

$$B_1 = \neg a, .$$

**We obtain** that

$$\neg a \vdash \neg a$$

also by the Deduction Theorem and assumption  $\vdash (A \Rightarrow A)$  in  $S$ .

**This proves** that So the holds for  $n = 0$

**Now assume** that the lemma holds for any  $A$  with  $j < n$  connectives.

**Prove:** lemma holds for  $A$  with  $n$  connectives.

**There are several** subcases to deal with.

**Case:**  $A$  is  $\neg A_1$

If  $A$  is of the form  $\neg A_1$  then  $A_1$  has less than  $n$  connectives.

**By the inductive assumption** we have the formulas

$$A'_1, B_1, B_2, \dots, B_n$$

corresponding to the  $A_1$  and the propositional variables  $b_1, b_2, \dots, b_n$  in  $A_1$ , such that

$$B_1, B_2, \dots, B_n \vdash A'_1$$

**Observe**, that the formulas  $A$  and  $\neg A_1$  have the same propositional variables.

**So the corresponding** formulas  $B_1, B_2, \dots, B_n$  are the same for both of them.

**We are going to show** that the inductive assumption allows us to prove that the lemma holds for  $A$ , ie. that

$$B_1, B_2, \dots, B_n \vdash A'.$$

**There two cases** to consider.

**Case:**  $v^*(A_1) = T$

**If**  $v^*(A_1) = T$  then by definition

$$A'_1 = A_1$$

and by the inductive assumption

$$B_1, B_2, \dots, B_n \vdash A_1$$

.

**In this case:**  $v^*(A) = v^*(\neg A_1) = \neg v^*(T) = F$

**So we have that**  $A' = \neg A = \neg\neg A_1$ .

**Since we have assumed** about  $S$  that

$$\vdash (A_1 \Rightarrow \neg\neg A_1)$$

**we obtain** by the monotonicity that also

$$B_1, B_2, \dots, B_n \vdash (A_1 \Rightarrow \neg\neg A_1).$$

**By inductive assumption** and Modus Ponens we have that also

$$B_1, B_2, \dots, B_n \vdash \neg\neg A_1,$$

that is

$$B_1, B_2, \dots, B_n \vdash \neg A,$$

that is

$$B_1, B_2, \dots, B_n \vdash A'.$$

**Case:**  $v^*(A_1) = F$

If  $v^*(A_1) = F$  then  $A'_1 = \neg A_1$  and  $v^*(A) = T$  so

$$A' = A.$$

**Therefore** the inductive assumption we have that  $B_1, B_2, \dots, B_n \vdash \neg A_1$ , that is (as  $A = \neg A_1$ )

$$B_1, B_2, \dots, B_n \vdash A'.$$

**Case:**  $A$  is  $(A_1 \Rightarrow A_2)$

If  $A$  is of the form  $(A_1 \Rightarrow A_2)$  then  $A_1$  and  $A_2$  have less than  $n$  connectives and so by the inductive assumption we have  $B_1, B_2, \dots, B_n \vdash A_1'$  and  $B_1, B_2, \dots, B_n \vdash A_2'$ , where  $B_1, B_2, \dots, B_n$  are formulas corresponding to the propositional variables in  $A$ . Here we have the following subcases to consider.

**Case:**  $v^*(A_1) = v^*(A_2) = T$

If  $v^*(A_1) = T$  then  $A_1'$  is  $A_1$  and if  $v^*(A_2) = T$  then  $A_2'$  is  $A_2$ . We also have  $v^*(A_1 \Rightarrow A_2) = T$  and so  $A'$  is  $(A_1 \Rightarrow A_2)$ . By the above and the inductive assumption, therefore,  $B_1, B_2, \dots, B_n \vdash A_2$  and since we have assumed about  $S$  that  $\vdash (A_2 \Rightarrow (A_1 \Rightarrow A_2))$ , we have by monotonicity and Modus Ponens, that  $B_1, B_2, \dots, B_n \vdash (A_1 \Rightarrow A_2)$ , that is  $B_1, B_2, \dots, B_n \vdash A'$ .

**Case:**  $v^*(A_1) = T, v^*(A_2) = F$

If  $v^*(A_1) = T$  then  $A_1'$  is  $A_1$  and if  $v^*(A_2) = F$  then  $A_2'$  is  $\neg A_2$ . Also we have in this case  $v^*(A_1 \Rightarrow A_2) = F$  and so  $A'$  is  $\neg(A_1 \Rightarrow A_2)$ . By the above and the inductive assumption, therefore,  $B_1, B_2, \dots, B_n \vdash \neg A_2$ . Since we have assumed ?? i.e.  $\vdash (A_1 \Rightarrow (\neg A_2 \Rightarrow \neg(A_1 \Rightarrow A_2)))$ , we have by monotonicity and Modus Ponens twice, that  $B_1, B_2, \dots, B_n \vdash \neg(A_1 \Rightarrow A_2)$ , that is  $B_1, B_2, \dots, B_n \vdash A'$ .

**Case:**  $v^*(A_1) = F$

If  $v^*(A_1) = F$  then  $A_1'$  is  $\neg A_1$  and, whatever value  $v$  gives  $A_2$ , we have  $v^*(A_1 \Rightarrow A_2) = T$  and so  $A'$  is  $(A_1 \Rightarrow A_2)$ . Therefore,  $B_1, B_2, \dots, B_n \vdash \neg A_1$  and since by ?? we have  $\vdash (\neg A_1 \Rightarrow (A_1 \Rightarrow A_2))$ , by monotonicity and Modus Ponens we get that  $B_1, B_2, \dots, B_n \vdash (A_1 \Rightarrow A_2)$ , that is  $B_1, B_2, \dots, B_n \vdash A'$ .

**With that we have covered all cases** and, by induction on  $n$ , the proof of the lemma is complete.

## **Proof of the Completeness Theorem**

**Assume** that  $\models A$ .

**Let**  $b_1, b_2, \dots, b_n$  be all propositional variables that occur in  $A$ .

**By the lemma** we know that, for any variable assignment  $v$ , the corresponding formulas  $A', B_1, B_2, \dots, B_n$  can be found such that

$$B_1, B_2, \dots, B_n \vdash A'$$

.

**Note here** that  $A'$  of the definition is  $A$  for any  $v$  since  $\models A$ .

**Hence**, if  $v$  is such that  $v(b_n) = T$ , then  $B_n$  is  $b_n$  and

$$B_1, B_2, \dots, b_n \vdash A.$$

**If**  $v(b_n) = F$ , then  $B_n$  is  $\neg b_n$  and by the lemma

$$B_1, B_2, \dots, \neg b_n \vdash A.$$

**So, by the Deduction Theorem**, we have

$$B_1, B_2, \dots, B_{n-1} \vdash (b_n \Rightarrow A)$$

and

$$B_1, B_2, \dots, B_{n-1} \vdash (\neg b_n \Rightarrow A).$$

**By monotonicity** and  $\vdash_S ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$

**we have that**

$$B_1, B_2, \dots, B_{n-1} \vdash ((b_n \Rightarrow A) \Rightarrow ((\neg b_n \Rightarrow A) \Rightarrow A)).$$

**Applying** Modus Ponens twice we get that

$$B_1, B_2, \dots, B_{n-1} \vdash A.$$

**Similarly,**  $v^*(B_{n-1})$  may be T or F, and, again applying Deduction Theorem, monotonicity, and  $\vdash_S ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$ , and Modus Ponens twice we can eliminate  $B_{n-1}$  just as we eliminated  $B_n$ .

**After n steps,** we finally obtain  $\vdash A$ .