Chapter 9
Completeness Theorem: Proof 1

We consider a sound proof system (under classical semantics)

$$S = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\quad \mathcal{AL},\quad MP\ ),$$

such that the formulas listed below are provable in S.

1. $(A \Rightarrow (B \Rightarrow A))$,

2. $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

3. $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$,

4. $(A \Rightarrow A)$,

5. $(B \Rightarrow \neg\neg B)$,

6. $(\neg A \Rightarrow (A \Rightarrow B))$,

7. $(A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$,

8. $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$,

9. $((\neg A \Rightarrow A) \Rightarrow A)$,

We present here two proofs of the following theorem.

Completeness Theorem For any formula A of S,

$$\models A \quad \text{if and only if} \quad \vdash_S A.$$

OBSERVATION 1 All the above formulas have proofs in the system H_2 and the system H_2 is sound, hence the Completeness Theorem for the system S implies the completeness of the system H_2.

OBSERVATION 2 We have assumed that the system S is sound, i.e. that the following theorem holds for S.

Soundness Theorem

For any formula A of S,

$$\text{if } \vdash_S A, \quad \text{then} \models A.$$

It means that in order to prove the Completeness Theorem we need to prove only the following implication.

For any formula A of S,

$$\text{If } \models A, \text{ then } \vdash_S A.$$

Both proofs of the Completeness Theorem relay strongly of the Deduction Theorem, as discussed and proved in the previous chapter.

Deduction theorem was proved for the system H_1 that is different that S, but all formulas that were used in its proof are provable in S, so it is valid for S as well, as it was for the system H_2, i.e. the following theorem holds.

Deduction Theorem for S

For any formulas A, B of S and Γ be any subset of formulas of S,

$$\Gamma,\ A \vdash_S B \ \text{ if and only if } \ \Gamma \vdash_S (A \Rightarrow B).$$

It is possible to prove the Completeness Theorem independently from the Deduction Theorem and we will present two of such a proof in later chapters.

The first proof presented here is similar in its structure to the proof of the deduction theorem and is due to **Kalmar, 1935.**

It shows how one can use the assumption that a formula A is a tautology in order to construct its formal proof. It is hence called **a proof - construction method**.

The second proof is a proof of the equivalent opposite implication to the Completeness part, i.e. we show how one can deduce that a formula A is not a tautology from the fact that it does not have a proof. It is hence called **a counter-model construction method**.

Completeness Theorem

A Proof - Construction Method

We first present one definition and to prove one lemma.

We write $\vdash A$ instead of $\vdash_S A$, as the system S is fixed.

Definition Let A be a formula and $b_1, b_2, ..., b_n$ be all propositional variables that occur in A.

Let v be variable assignment $v : VAR \longrightarrow \{T, F\}$.

DEFINITION 1

We define, for $A, b_1, b_2, ..., b_n$ and v a corresponding formulas A', $B_1, B_2, ..., B_n$ as follows:

$$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases}$$

$$B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$$

for $i = 1, 2, ..., n$.

Example 1: let A be a formula $(a \Rightarrow \neg b)$.

Let v be such that

$$v(a) = T, \quad v(b) = F.$$

In this case: $b_1 = a$, $b_2 = b$, and $v^*(A) = v^*(a \Rightarrow \neg b) = v(a) \Rightarrow \neg v(b) = T \Rightarrow \neg F = T$.

The corresponding A', B_1, B_2 are:

$A' = A$ (as $v^*(A) = T$),

$B_1 = a$ (as $v(a) = T$),

$B_2 = \neg b$ (as $v(b) = F$).

Example 2

Let A be a formula

$$((\neg a \Rightarrow \neg b) \Rightarrow c)$$

and let v be such that

$$v(a) = T, \quad v(b) = F, v(c) = F.$$

Evaluate A', $B_1, ...B_n$ as defined by the definition 1.

In this case $n = 3$ and

$$b_1 = a,\ b_2 = b, b_3 = c,$$

and we evaluate

$$v^*(A) = v^*((\neg a \Rightarrow \neg b) \Rightarrow c) =$$

$$((\neg v(a) \Rightarrow \neg v(b)) \Rightarrow v(c)) =$$

$$((\neg T \Rightarrow \neg F) \Rightarrow F) = (T \Rightarrow F) = F.$$

The corresponding A', B_1, B_2, B_2 are:

$$A' = \neg A = \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$$

as $v^*(A) = F$,

$B_1 = a$ (as $v(a) = T$),

$B_2 = \neg b$ (as $v(b) = F$).

$B_3 = \neg c$ (as $v(c) = F$).

The lemma stated below describes a method of transforming a semantic notion of a tautology into a syntactic notion of provability. It defines, for any formula A and a variable assignment v a corresponding deducibility relation.

MAIN LEMMA For any formula A and a variable assignment v, if $A^{'}$, B_1 , B_2, ..., B_n are corresponding formulas defined by our definition, then

$$B_1, B_2, ..., B_n \;\vdash\; A'.$$

Example 3 Let A, v be as defined by the example 1, then the Lemma asserts that

$$a, \neg b \ \vdash \ (a \Rightarrow \neg b).$$

Example 4 Let A, v be as defined in example 2, then the lemma asserts that

$$a, \neg b, \neg c \ \vdash \ \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$$

Proof of the MAIN LEMMA The proof is by induction on the degree of A i.e. a number n of logical connectives in A.

Case: $n = 0$

In the case that $n = 0$ A is atomic and so consists of a single propositional variable, say a.

Clearly, if $v^*(A) = T$ then we $A' = A = a$, $B_1 = a$.

We obtain that

$$a \vdash a$$

by the Deduction Theorem and the fact that $\vdash (A \Rightarrow A)$, i.e. also $\vdash (a \Rightarrow a)$.

In case when $v^*(A) = F$ we have that

$$A' = \neg A = \neg a,$$

$$B_1 = \neg a, .$$

We obtain that

$$\neg a \ \vdash \ \neg a$$

also by the Deduction Theorem and assumption $\vdash (A \Rightarrow A)$ in S.

This proves that Lemma holds for $n = 0$

Now assume that the lemma holds for any A with $j < n$ connectives.

Prove: lemma holds for A with n connectives.

There are several subcases to deal with.

Case: A **is** $\neg A_1$

If A is of the form $\neg A_1$ then A_1 has less then n connectives.

By the inductive assumption we have the formulas

$$A_1^{'}, \quad B_1, B_2, ..., B_n$$

corresponding to the A_1 and the propositional variables $b_1, b_2, ..., b_n$ in A_1, such that

$$B_1, B_2, ..., B_n \ \vdash \ A_1^{'}$$

Observe, that the formulas A and $\neg A_1$ have the same propositional variables.

So the corresponding formulas B_1 , B_2, ..., B_n are the same for both of them.

We are going to show that the inductive assumption allows us to prove that the lemma holds for A, ie. that

$$B_1, B_2, ..., B_n \ \vdash \ A^{'}.$$

There two cases to consider.

Case: $v^*(A_1) = T$

If $v^*(A_1) = T$ then by definition

$$A_1^{'} = A_1$$

and by the inductive assumption

$$B_1, B_2, ..., B_n \ \vdash \ A_1$$

.

In this case: $v^*(A) = v^*(\neg A_1) = \neg v^*(T) = F$

So we have that $A' = \neg A = \neg\neg A_1$.

Since we have assumed about S that

$$\vdash \ (A_1 \Rightarrow \neg\neg A_1)$$

we obtain by the monotonicity that also

$$B_1, B_2, ..., B_n \ \vdash \ (A_1 \Rightarrow \neg\neg A_1).$$

By inductive assumption and Modus Ponens we have that also

$$B_1, B_2, ..., B_n \vdash \neg\neg A_1,$$

and as $A^{'} = \neg A = \neg\neg A_1$ we get

$$B_1, B_2, ..., B_n \vdash \neg A,$$

$$B_1, B_2, ..., B_n \vdash A^{'}.$$

Case: $v^*(A_1) = F$

If $v^*(A_1) = F$ then $A_1^{'} = \neg A_1$ and $v^*(A) = T$ so $A^{'} = A.$

Therefore by the inductive assumption we have that

$$B_1, B_2, ..., B_n \vdash \neg A_1$$

that is (as $A = \neg A_1$)

$$B_1, B_2, ..., B_n \vdash A^{'}.$$

Case: A **is** $(A_1 \Rightarrow A_2)$

If A is $(A_1 \Rightarrow A_2)$ then A_1 and A_2 have less than n connectives.

By the inductive assumption we have
$B_1, B_2, ..., B_n \vdash A_1{}'$

and
$B_1, B_2, ..., B_n \vdash A_2{}'$,

where $B_1, B_2, ..., B_n$ are formulas corresponding to the propositional variables in A.

Now we have the following subcases to consider.

Case: $v^*(A_1) = v^*(A_2) = T$

If $v^*(A_1) = T$ then $A_1{}'$ is A_1 and if $v^*(A_2) = T$ then $A_2{}'$ is A_2.
We also have $v^*(A_1 \Rightarrow A_2) = T$ and so A' is $(A_1 \Rightarrow A_2)$.

By the above and the inductive assumption, therefore,
$B_1, B_2, ..., B_n \vdash A_2$ and since we have assumed about S that $\vdash (A_2 \Rightarrow (A_1 \Rightarrow A_2))$,

we have by monotonicity and Modus Ponens, that $B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow A_2)$, that is

$$B_1, B_2, ..., B_n \vdash A'.$$

Case: $v^*(A_1) = T, v^*(A_2) = F$

If $v^*(A_1) = T$ then ${A_1}' = A_1$ and

if $v^*(A_2) = F$ then ${A_2}' = \neg A_2$.

Also we have in this case $v^*(A_1 \Rightarrow A_2) = F$ and so $A' = \neg(A_1 \Rightarrow A_2)$.

By the above and the inductive assumption, therefore, $B_1, B_2, ..., B_n \vdash \neg A_2$. Since we have assumed $\vdash (A_1 \Rightarrow (\neg A_2 \Rightarrow \neg(A_1 \Rightarrow A_2)))$, we have by monotonicity and Modus Ponens twice, that $B_1, B_2, ..., B_n \vdash \neg(A_1 \Rightarrow A_2)$, that is

$$B_1, B_2, ..., B_n \vdash A'.$$

Case: $v^*(A_1) = F$

If $v^*(A_1) = F$ then ${A_1}^{'} = \neg A_1$ and, whatever value v gives A_2, we have that $v^*(A_1 \Rightarrow A_2) = T$ and so $A^{'} = (A_1 \Rightarrow A_2)$.

Therefore,

$$B_1, B_2, ..., B_n \ \vdash \ \neg A_1$$

and since $\vdash \ (\neg A_1 \Rightarrow (A_1 \Rightarrow A_2))$, by monotonicity and Modus Ponens we get that

$$B_1, B_2, ..., B_n \ \vdash \ (A_1 \Rightarrow A_2),$$

that is

$$B_1, B_2, ..., B_n \ \vdash \ A^{'}.$$

With that we have covered all cases and, by induction on n, the proof of the lemma is complete.

Proof of the Completeness Theorem

Assume that $\models A$.

Let $b_1, b_2, ..., b_n$ be all propositional variables that occur in A, i.e. $A = A(b_1, b_2, ..., b_n)$.

By the lemma we know that, for any variable assignment v, the corresponding formulas $A^{'}$, B_1 , B_2, ..., B_n can be found such that

$$B_1, B_2, ..., B_n \ \vdash \ A^{'}$$

.

Note here that $A^{'}$ of the definition is A for any v since $\models A$.

Hence, if v is such that $v(b_n) = T$, then B_n is b_n and

$$B_1, B_2, ..., b_n \ \vdash \ A.$$

If w is such that $w(b_n) = F$, then B_n is $\neg b_n$ and by the lemma

$$B_1, B_2, ..., \neg b_n \ \vdash \ A.$$

So, by the Deduction Theorem, we have

$$B_1, B_2, ..., B_{n-1} \ \vdash \ (b_n \Rightarrow A)$$

and

$$B_1, B_2, ..., B_{n-1} \ \vdash \ (\neg b_n \Rightarrow A).$$

By monotonicity and assumed formula 9

$$\vdash_S ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$$

we have that $B_1, B_2, ..., B_{n-1} \vdash ((b_n \Rightarrow A) \Rightarrow ((\neg b_n \Rightarrow A) \Rightarrow A))$.

Applying Modus Ponens twice we get that $B_1, B_2, ..., B_{n-1} \vdash A$.

Similarly, $v^*(B_{n-1})$ may be T or F, and, again applying Deduction Theorem, monotonicity, and $\vdash_S ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$, and Modus Ponens twice we can eliminate B_{n-1} just as we eliminated B_n.

After n steps, we finally obtain proof of A in S, i.e. we have that

$$\vdash A.$$