Chapter 9

Completeness Theorem (Part 1) Proof 1 and Examples

We consider a sound proof system (under classical semantics)

$$S = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\quad \mathcal{AX},\quad MP\ ),$$

such that the formulas listed below are provable in S.

1. $(A \Rightarrow (B \Rightarrow A))$,

2. $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

3. $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$,

4. $(A \Rightarrow A)$,

5. $(B \Rightarrow \neg\neg B)$,

6. $(\neg A \Rightarrow (A \Rightarrow B))$,

7. $(A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$,

8. $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$,

9. $((\neg A \Rightarrow A) \Rightarrow A)$.

Deduction Theorem for S

For any formulas A, B of S and Γ be any subset of formulas of S,

$$\Gamma,\ A \vdash_S B \quad \text{if and only if} \quad \Gamma \vdash_S (A \Rightarrow B).$$

Completeness Theorem for S

For any formula A of S,

$$\models\ A \quad \text{if and only if} \quad \vdash_S A.$$

MAIN DEFINITION for Proof 1

We define, for any $A(b_1, b_2, ..., b_n)$ and any v a corresponding formulas A', $B_1, B_2, ..., B_n$ as follows:

$$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases}$$

$$B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$$

for $i = 1, 2, ..., n$.

MAIN LEMMA for Proof 1

For any formula A and a truth assignment v, if $A^{'}$, B_1 , B_2, ..., B_n are corresponding formulas defined by the Main Definition, then

$$B_1, B_2, ..., B_n \ \vdash \ A'.$$

PROOF 1 of the Completeness Theorem

Assume that $\models A$.

Let $b_1, b_2, ..., b_n$ be all propositional variables that occur in A, i.e. $A = A(b_1, b_2, ..., b_n)$.

By Main Lemma we know that, for any variable assignment v, the corresponding formulas $A^{'}$, B_1 , B_2, ..., B_n can be found such that

$$B_1, B_2, ..., B_n \ \vdash \ A^{'}$$

.

Note here that $A^{'}$ of the Main Definition is A for any v, since $\models A$, i.e.

$$B_1, B_2, ..., B_n \ \vdash \ A.$$

The proof is based on a method of constructive elimination of all hypothesis $B_1, B_2, ..., B_n$ to finally show that A has a proof in S without them, i.e. $\vdash\ A$.

Step 1: elimination of B_n.

We have 2 cases to consider.

Case 1: let v be such that $v(b_n) = T$. Then $B_n = b_n$ and we have that

$$B_1, B_2, ..., b_n \ \vdash\ A.$$

By Deduction Theorem, we have that

$$B_1, B_2, ..., B_{n-1} \ \vdash\ (b_n \Rightarrow A).$$

Case 2: let be such that $v(b_n) = F$.
Then $B_n = \neg b_n$ and by the lemma

$$B_1, B_2, ..., \neg b_n \ \vdash \ A.$$

By the Deduction Theorem, we have that

$$B_1, B_2, ..., B_{n-1} \ \vdash \ (\neg b_n \Rightarrow A).$$

By the assumed formula 9

$$\vdash ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$$

for $A = b_n, B = A$ and monotonicity we have

$$B_1, B_2, ..., B_{n-1} \vdash ((b_n \Rightarrow A) \Rightarrow ((\neg b_n \Rightarrow A) \Rightarrow A)).$$

Applying Modus Ponens twice to the above and Case 1, Case 2 we get that

$$B_1, B_2, ..., B_{n-1} \vdash A.$$

End of B_n elimination.

Step 2: elimination of B_{n-1}.

We repeat the Step 1. As before, $v^*(B_{n-1})$ may be T or F, and, applying Main Lemma, Deduction Theorem, monotonicity, proper substitutions of assumed formula 9 $\vdash ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$, and Modus Ponens twice we can eliminate B_{n-1} just as we eliminated B_n.

After n steps, we finally obtain that

$$\vdash A.$$

Observe that our proof of the fact that $\vdash A$ is a constructive one. Moreover, we have used in it only Main Lemma and Deduction Theorem which both have a constructive proofs.

We can hence reconstruct proofs in each case when we apply these theorems back to the original axioms $A1 - A3$ of H_2. The same applies to the proofs in H_2 of all formulas 1 -9 of the system S.

It means that for any A, such that $\models A$, each v restricted to A provides us the method of a construction of the formal proof of A in H_2, or in any system S in which formulas 1 -9 are provable.

EXAMPLE As an example of how the Completeness Theorem proof works, we consider the case in which A is a tautology

$$(a \Rightarrow (\neg a \Rightarrow b))$$

and show how the construction described in the Proof 1 works; i.e how we construct the proof of A.

Step 1. We apply Main Lemma to all different variable assignments for A. We have 4 cases to consider. As $\models A$ in all cases we have that $A' = A$.

Case 1: $v(a) = T, v(b) = T$.
In this case $B_1 = a, B_2 = b$ and, as in all cases $A' = A$.

By the Main Lemma,

$$a, b \vdash (a \Rightarrow (\neg a \Rightarrow b)).$$

Case 2: $v(a) = T, v(b) = F$.

In this case $B_1 = a, B_2 = \neg b$, $A^{'} = A$ and by the Main Lemma,

$$a, \neg b \vdash (a \Rightarrow (\neg a \Rightarrow b)).$$

Case 3: $v(a) = F, v(b) = T$.

In this case $B_1 = \neg a, B_2 = b$, $A^{'} = A$ and by the Main Lemma,

$$\neg a, b \vdash (a \Rightarrow (\neg a \Rightarrow b)).$$

Case 4: $v(a) = F, v(b) = F$.

In this case $B_1 = \neg a, B_2 = \neg b$, $A^{'} = A$ and by the Main Lemma,

$$\neg a, \neg b \vdash (a \Rightarrow (\neg a \Rightarrow b)).$$

We apply Deduction Theorem on formulas $b, \neg b$ to all the cases 1-3. This is the case of B_n elimination in the Proof 1.

D1 (Cases 1 and 2)

$$a \ \vdash \ (b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))),$$

$$a \ \vdash \ (\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))),$$

D2 (Cases 2 and 3)

$$\neg a \ \vdash \ (b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))),$$

$$\neg a \ \vdash \ (\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))).$$

By the monotonicity and proper substitution of the formula 9 we have that

$a \vdash ((b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow ((\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow (a \Rightarrow (\neg a \Rightarrow b)))$,

$\neg a \vdash ((b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow ((\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow (a \Rightarrow (\neg a \Rightarrow b)))$.

Applying Modus Ponens twice to **D1**, **D2** and these above, respectively, gives us

$$a \vdash (a \Rightarrow (\neg a \Rightarrow b)) \text{ and}$$

$$\neg a \vdash (a \Rightarrow (\neg a \Rightarrow b)).$$

Applying the Deduction Theorem to the above we obtain

D3 $\vdash (a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b)))$ and

D4 $\vdash (\neg a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b)))$.

Applying Modus Ponens twice to **D3** and **D4** and the following form of formula 9,

$$\vdash ((a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow ((\neg a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))))$$

we get finally that

$$\vdash (a \Rightarrow (\neg a \Rightarrow b)).$$