CSE371 PRACTICE Q1 SOLUTIONS

QUESTION 1 Give a definition and an example of a default reasoning.

Default reasoning is a reasoning in which it is allowed to draw plausible inferences from less-then-conclusive evidence in the absence of information to the contrary.

Example: Consider a statement *Birds fly.* Tweety, we are told, is a bird. From this, and the fact that birds fly, we conclude that Tweety can fly.

This conclusion, however is *defeasible*: Tweety may be an ostrich, a penguin, a bird with a broken wing, or a bird whose feet have been set in concrete. But as long as we don't have the evidence to the contrary (*Tweedy has a broken wing*) we accept the conclusion that *Tweety can fly.*

QUESTION 2 Write the following natural language statement:

From the fact that it is not necessary that an elephant is not a bird we deduce that:
it is not possible that an elephant is a bird or, if it is possible that an elephant is a bird, then it is not necessary that a bird flies.

as a formula

(i) $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \mathbf{C}, \mathbf{I}, \cap, \cup, \Rightarrow\}}$, and as a formula

(ii) $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$.

Solution

(i) We translate our statement into a formula

$A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \mathbf{C}, \mathbf{I}, \cap, \cup, \Rightarrow\}}$ as follows.

Propositional Variables: a, b.
a denotes statement: *an elephant is a bird*, b denotes a statement: *a bird flies.*

Propositional Modal Connectives: C, I.
C denotes statement: *it is possible that*, **I** denotes statement: *it is necessary that.*

Translation 1:

$$A_1 = (\neg \mathbf{I} \neg a \Rightarrow (\neg \mathbf{C} a \cup (\mathbf{C} a \Rightarrow \neg \mathbf{I} b))).$$

(ii) Now we translate our statement into a formula

$A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$ as follows.

Propositional Variables: a, b, c.

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a denotes statement: *it is necessary that an elephant is not a bird* ,
b denotes statement: *it is possible that an elephant is a bird* ,
c denotes a statement: *it is necessary that a bird flies.*

Translation 2:

$$A_2 = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))).$$

2. Main connective of the formula A_1 is: $\Rightarrow$, main connective of the formula A_2 is also $\Rightarrow$.

3. Degree of the formula A_1 is: 11, degree of the formula A_2 is: 6.

4. All proper, non-atomic sub-formulas of A_1 are:

$$\neg \mathbf{I}\neg a, (\neg \mathbf{C}a \cup (\mathbf{C}a \Rightarrow \neg \mathbf{I}b)), \mathbf{I}\neg a, \neg a, \neg \mathbf{C}a, (\mathbf{C}a \Rightarrow \neg \mathbf{I}b), \mathbf{C}a, \neg \mathbf{I}b, \mathbf{I}b$$

5. All non-atomic sub-formulas of A_2 are:

$$(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)), \neg a, (\neg b \cup (b \Rightarrow \neg c)), \neg b, (b \Rightarrow \neg c), \neg c$$

6. Find a model and a counter-model restricted to A_2. Use short-hand notation. Show work.

A restricted model: $a = T, b = T, c = F$

Evaluation: $(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)) = T$ for, for example $a = T$ and b, c any truth values. ($F \Rightarrow$ *anything* $= T$).

$a = T$ gives 4 models (2^2 values on b and c.)

A Restricted counter-model: $a = F, b = T$ and $c = T$

Evaluation: $(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c)) = F$ if and only if
$\neg a = T$ and $(\neg b \cup (b \Rightarrow \neg c)) = F$, iff
$a = F, \neg b = F$ and $(b \Rightarrow \neg c) = F$, iff
$a = F, b = T$ and $(T \Rightarrow \neg c) = F$, iff
$a = F, b = T$ and $\neg c = F$ iff
$a = F, b = T$ and $c = T$

7. Statement: *There are more then 3 possible restricted counter-models of* A_2*.* is not true. There is only one possible counter-model restricted to A_2 as shown by above evaluation.

8. Statement: *There are more then 2 possible restricted models of* A_2*.* is true. There are 7 possible restricted models of A_2. Justification: $2^3 - 1 = 7$.

9. List 3 models and 3 counter-models for A_2 by extending the model and the counter-model you have found in **5.** to the VAR of all variables.

A model for A_2 is, by definition, any function

$$w : VAR \longrightarrow \{T, F\},$$

such that $w(A_2) = T$.

A restricted model for A_2 is, as defined in **7.** is a function

$$v : \{a, b, c\} \longrightarrow \{T, F\},$$

such that $v(A_2) = T$, i.e. for example:
$A = T, b = T, c = F$.

We extend v to the set of all propositional variables VAR to obtain a (non restricted) model. Here are three of such extensions.

Model w_1

$$w_1 : VAR \longrightarrow \{T, F\}$$

$$w_1(a) = v(a) = T,\ \ w_1(b) = v(b) = T, w_1(c) = v(c) = F,\ \ and\, w_1(x) = T,\ \ for\ \ all\ \ x \in VAR - \{a, b, c\}.$$

Model w_2:

$$w_2(a) = v(a) = T,\ \ \ w_2(b) = v(b) = T, w_2(c) = v(c) = F,\ \ and\ w_2(x) = F,$$
$$for\ \ \ all\ \ \ x \in VAR - \{a, b, c\}.$$

Model w_3:

$$w_3(a) = v(a) = T,\ \ w_3(b) = v(b) = T,\ \ w_3(c) = v(c) = F, w_3(d) = Fand\ w_3(x) = T,$$
$$for\ \ \ all\ \ \ x \in VAR - \{a, b, c, d\}.$$

There is an many of such models, as extensions of v to the set VAR, i.e. as many as real numbers.

A counter-model for A_2, by definition, is any function

$$w : VAR \longrightarrow \{T, F\},$$

such that $w(A_2) = F$.

A restricted counter-model for A_2 is, as defined in **6.** a function

$$v : \{a, b\} \longrightarrow \{T, F\},$$

such that $v(A) = F$, i.e. (only one) such that $v(a) = F,\ v(b) = T,\ v(c) = T$.

There is only one **restricted counter-model** v for A_2.

We extend v to the set of all propositional variables VAR to obtain a (non restricted) counter-models. Here are three of such extensions.

Counter- model w_1:

$$w_1(a) = v(a) = F,\ \ w_1(b) = v(b) = T, w_1(c) = v(c) = T,\ \ and\, w_1(x) = F,\ \ for\ \ all\ \ x \in VAR - \{a, b, c\}.$$

Counter- model w_2:

$$w_2(a) = v(a) = T,\ w_2(b) = v(b) = T,\ w_2(c) = v(c) = T,\ and\ w_2(x) = T,$$

$$for\ \ all\ \ x \in VAR - \{a, b, c\}.$$

There is an many of such counter- models, as extensions of v to the set VAR, i.e. as many as real numbers.

9. There are $2^{\aleph_0} = \mathcal{C}$ possible models for A_2. There are $2^{\aleph_0} = \mathcal{C}$ possible counter-models for A_2.

QUESTION 3 Show that

$$\models (\neg((a \cap \neg b) \Rightarrow ((c \Rightarrow (\neg f \cup d)) \cup e)) \Rightarrow ((a \cap \neg b) \cap (\neg(c \Rightarrow (\neg f \cup d)) \cap \neg e))).$$

Observe that $VAR_A = \{a, b, c, d, e, f\}$, so there are $2^6 = 64$ truth assignments to consider. Much too many to apply the truth table method.

The "proof by contradiction" method may be shorter, but before we apply it let's look closer at the sub-formulas of A and patterns they form inside the formula A, i.e. we apply the substitution method first.
We denote : $B = (a \cap \neg b), C = (c \Rightarrow (\neg f \cup d)$, and $D = e$. We re-write A as

$$(\neg(B \Rightarrow (C \cup D)) \Rightarrow (B \cap (\neg C \cap \neg D))).$$

Now we apply "proof by contradiction" method.

Step 1: Assume $(\neg(B \Rightarrow (C \cup D)) \Rightarrow (B \cap (\neg C \cap \neg D))) = F$. It is possible **only** when $(B \Rightarrow (C \cup D)) = F$ and $(B \cap (\neg C \cap \neg D)) = F$.

Step 2: $(B \Rightarrow (C \cup D)) = F$ **only** when

$$B = T, C = F, D = F.$$

Step 3: From **Step 1** we have that

$$(B \cap (\neg C \cap \neg D)) = F.$$

We now evaluate its logical value for $B = T, C = F, D = F$ obtained in **Step 2**, i.e. compute:

$$(T \cap (\neg F \cap \neg F)) = F,$$

$$(T \cap (T \cap T)) = F,$$

$$T = F.$$

Contradiction. This proves that

$$\models (\neg(B \Rightarrow (C \cup D)) \Rightarrow (B \cap (\neg C \cap \neg D))),$$

and hence

$$\models (\neg((a \cup b) \Rightarrow ((c \Rightarrow d) \cup e)) \Rightarrow ((a \cup b) \cap (\neg(c \Rightarrow d) \cap \neg e))).$$

All truth assignments are models for A, i.e. A does not have a counter-model.

REMINDER: We define **H** semantics operations $\cup$ and $\cap$ as follows

$$a \cup b = max\{a, b\}, \quad a \cap b = min\{a, b\}.$$

The Truth Tables for Implication and Negation are:

H-Implication

$\Rightarrow$	F	$\bot$	T
F	T	T	T
$\bot$	F	T	T
T	F	$\bot$	T

H Negation

$\neg$	F	$\bot$	T
	T	F	F

QUESTION 4 **We know that**

$$v : VAR \longrightarrow \{F, \bot, T\}$$

is such that

$$v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \bot$$

under **H** semantics. **evaluate** $v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))$.

Solution : $v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \bot$ under H semantics if and only if (we use shorthand notation) $(a \cap b) = T$ and $(a \Rightarrow c) = \bot$ if and only if $a = T, b = T$ and $(T \Rightarrow c) = \bot$ if and only i f $c = \bot$. I.e. we have that

$$v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \bot \quad iff \quad a = T,\ b = T, c = \bot\ .$$

Now we can we **evaluate** $v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))$ as follows (in shorthand notation).
$v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b)) =$
$(((T \Rightarrow T) \Rightarrow (T \Rightarrow \neg \bot)) \cup (T \Rightarrow T)) =$
$((T \Rightarrow (T \Rightarrow F)) \cup T) = T$.

We define a 4 valued $\mathbf{Ł}_4$ logic semantics as follows. The language is

$$\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}.$$

We define the logical connectives $\neg, \Rightarrow, \cup, \cap$ of $\mathbf{Ł}_4$ as the following operations in the set $\{F, \bot_1, \bot_2, T\}$, where $\{F < \bot_1 < \bot_2 < T\}$.

Negation $\neg$

$$\neg : \quad \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \bot_1, \bot_2, T\},$$

such that

$$\neg\bot_1 = \bot_1, \quad \neg\bot_2 = \bot_2, \quad \neg F = T, \quad \neg T = F.$$

Conjunction $\cap$

$$\cap : \quad \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \ \bot_1, \bot_2, , T\}$$

such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$,

$$a \cap b = min\{a, b\}.$$

Disjunction $\cup$

$$\cup : \quad \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \bot_1, \bot_2, T\}$$

such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$,

$$a \cup b = max\{a, b\}.$$

Implication $\Rightarrow$

$$\Rightarrow : \quad \{F, \bot_1, \bot_2, T\} \times \{F, \bot_1, \bot_2, T\} \longrightarrow \{F, \bot_1, \bot_2, T\},$$

such that for any $a, b \in \{F, \bot_1, \bot_2, T\}$,

$$a \Rightarrow b = \begin{cases} \neg a \cup b & \text{if } a > b \\ T & \text{otherwise} \end{cases}$$

QUESTION 5

Part 1 Write all TTables for **Ł**$_4$

Solution :

Ł$_4$ **Negation**

$\neg$	F	$\bot_1$	$\bot_2$	T
	T	$\bot_1$	$\bot_2$	F

Ł$_4$ **Conjunction**

$\cap$	F	$\bot_1$	$\bot_2$	T
F	F	F	F	F
$\bot_1$	F	$\bot_1$	$\bot_1$	$\bot_1$
$\bot_2$	F	$\bot_1$	$\bot_2$	$\bot_2$
T	F	$\bot_1$	$\bot_2$	T

Ł$_4$ **Disjunction**

$\cup$	F	$\bot_1$	$\bot_2$	T
F	F	$\bot_1$	$\bot_2$	T
$\bot_1$	$\bot_1$	$\bot_1$	$\bot_2$	T
$\bot_2$	$\bot_2$	$\bot_2$	$\bot_2$	T
T	T	T	T	T

Ł$_4$**-Implication**

$\Rightarrow$	F	$\bot_1$	$\bot_2$	T
F	T	T	T	T
$\bot_1$	$\bot_1$	T	T	T
$\bot_2$	$\bot_2$	$\bot_2$	T	T
T	F	$\bot_1$	$\bot_2$	T

Part 2 Verify whether

$$\models_{Ł_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

Solution : Let v be a truth assignment such that $v(a) = v(b) = \bot_1$.

We evaluate $v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\bot_1 \Rightarrow \bot_1) \Rightarrow (\neg \bot_1 \cup \bot_1)) = (T \Rightarrow (\bot_1 \cup \bot_1)) = (T \Rightarrow \bot_1) = \bot_1$.

This proves that v is a counter-model for our formula and

$$\not\models_{Ł_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b)).$$

Observe that a v such that $v(a) = v(b) = \bot_2$ is also a counter model, as $v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\bot_2 \Rightarrow \bot_2) \Rightarrow (\neg \bot_2 \cup \bot_2)) = (T \Rightarrow (\bot_2 \cup \bot_2)) = (T \Rightarrow \bot_2) = \bot_2$.