

CSE371 Q1 SOLUTIONS

QUESTION 1 Describe in few words Mc Carthy critique of Amarel solution to *Missionaries and Cannibal* puzzle.

Mc Carthy said that *The correctness of Amarel's representation (mathematically correct!) is not an ordinary logical consequence of the problem statement.* First, nothing has been said, for example about the properties of boats or even the fact that rowing across the river doesn't change the number of missionaries or cannibals or the capacity of the boat. Moreover, more things can happen! Imagine giving someone a problem, and after he puzzles for a while, he suggests going upstream half a mile and crossing on a bridge. *What a bridge?* you say. *No bridge is mentioned in the statement of the problem.* And this dunce replies, *Well, they don't say the isn't a bridge.* etc, etc..

QUESTION 2

1. Write the following natural language statement

If it is not believed that quiz is easy or quiz is not easy, then from the fact that $2 + 2 = 5$ we deduce that it is believed that quiz is easy.

as a formula

Formula 1 $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_1 = \mathcal{L}_{\{\neg, \mathbf{B}, \cup, \Rightarrow\}}$, where \mathbf{B} is a believe connective. Statement $\mathbf{B}A$ says: *It is believed that A.*

Translation:

Propositional Variables: a, b .

a denotes statement: *quiz is easy*,

b denotes a statement: $2 + 2 = 5$.

Propositional Believe Connective: \mathbf{B} .

$$A_1 = ((\neg \mathbf{B}a \cup \neg a) \Rightarrow (b \Rightarrow \mathbf{B}a)).$$

Alternative Translation: due to ambiguity of natural language understanding of strength (parenthesis) of logical connectives we can have the following alternative translation A .

$$A = (\neg \mathbf{B}(a \cup \neg a) \Rightarrow (b \Rightarrow \mathbf{B}a)).$$

Formula 2 $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_2 = \mathcal{L}_{\{\neg, \cup, \Rightarrow\}}$.

Translation:

Propositional Variables: a, b, c .

a denotes statement: *it is believed that quiz is easy*,

b denotes statement: *quiz is easy*, and
 c denotes a statement: $2 + 2 = 5$.

$$A_2 = ((\neg a \cup \neg b) \Rightarrow (c \Rightarrow a)).$$

2. Degree of the formula A_1 is: 7, degree of the formula A_2 is 5.
3. All proper sub-formulas of A_1 are:
 $(\neg \mathbf{B}a \cup \neg a)$, $(b \Rightarrow \mathbf{B}a)$, $\neg \mathbf{B}a$, $\neg a$, $\mathbf{B}a$, a , b
4. All non-atomic sub-formulas of A_2 are:
 $((\neg a \cup \neg b) \Rightarrow (c \Rightarrow a))$, $(\neg a \cup \neg b)$, $(c \Rightarrow a)$, $\neg a$, $\neg b$
5. Find all counter-models (restricted) for A_2 . Use short-hand notation. Don't construct Truth Tables! Explain.

Counter-Models evaluation: (we use shorthand notation).

$((\neg a \cup \neg b) \Rightarrow (c \Rightarrow a)) = F$ if and only if $(\neg a \cup \neg b) = T$ and $(c \Rightarrow a) = F$.

But $(c \Rightarrow a) = F$ if and only if $c = T$ and $a = F$, and we get from $(\neg a \cup \neg b) = T$ that $(\neg F \cup \neg b) = T \cup \neg b = T$. This is true for $b = T$ and $b = F$.

All Restricted Counter-Models are:

$a = F$, $b = T$, $c = T$, and $a = F$, $b = F$, $c = T$.

We write it formally as follows.

Restricted Counter model v_1 is a function

$$v_1 : \{a, b, c\} \longrightarrow \{T, F\},$$

such that $v_1(a) = F$, $v_1(b) = T$, $v_1(c) = T$.

Restricted Counter model v_2 is a function

$$v_2 : \{a, b, c\} \longrightarrow \{T, F\},$$

such that $v_2(a) = F$, $v_2(b) = F$, $v_2(c) = T$.

6. Find a model (restricted) for A_2 . Use short-hand notation. Don't construct Truth Tables! Explain.

Restricted Model is: $a = T$, $b = T$, $c = F$.

Verification: $((\neg T \cup \neg T) \Rightarrow (T \Rightarrow T)) = ((F \cup F) \Rightarrow T) = (F \Rightarrow T) = T$.

7. There are 6 possible models (restricted) for A_2 . Don't need to list them, just justify your answer: $2^3 - 2 = 6$.
8. List 2 models (not restricted) for A_2 by extending the model you have found in 6. to the VAR of all variables.

Model w_1 , extending restricted $v : \{a, b, c\} \longrightarrow \{T, F\}$, such that $v(a) = F$, $v(b) = T$, $v(c) = F$ to the set of variables VAR is defined as follows.

$w_1(a) = v(a) = T$, $w_1(b) = v(b) = T$, $w_1(c) = v(c) = F$ and $w_1(x) = T$, for all $x \in VAR - \{a, b, c\}$.

Model w_2 :

$$w_2(a) = v(a) = T, \quad w_2(b) = v(b) = T, \quad w_1(c) = v(c) = F \text{ and } w_2(x) = F, \quad \text{for all } x \in VAR - \{a, b, c\}.$$

9. There are \mathcal{C} (as many as real numbers) of possible models for A_2 . There are \mathcal{C} (as many as real numbers) of possible counter-models for A_2 .

QUESTION 3 Show that $v \models ((\neg a \cup b) \Rightarrow (((c \cap d) \Rightarrow \neg d) \Rightarrow (\neg a \cup b)))$ for all $v : VAR \rightarrow \{T, F\}$, i.e. that

$$\models ((\neg a \cup b) \Rightarrow (((c \cap d) \Rightarrow \neg d) \Rightarrow (\neg a \cup b))).$$

Solution: we apply first the substitution method. We substitute:

$$A = (\neg a \cup b), \quad B = ((c \cap d) \Rightarrow \neg d)$$

and our initial formula becomes $(A \Rightarrow (B \Rightarrow A))$.

As the second step we show that

$$\models (A \Rightarrow (B \Rightarrow A))$$

using "proof by contradiction" method. We use short hand notation.

Assume that $(A \Rightarrow (B \Rightarrow A)) = F$. This is possible only if $A = T$ and $(B \Rightarrow A) = F$. Substituting $A = T$ in $(B \Rightarrow A) = F$ we get $(B \Rightarrow T) = F$ what is impossible. This proves that $\models (A \Rightarrow (B \Rightarrow A))$ and by substitution theorem, also $\models ((\neg a \cup b) \Rightarrow (((c \cap d) \Rightarrow \neg d) \Rightarrow (\neg a \cup b)))$.