

CSE371 Q1 SOLUTIONS Fall 2011

QUESTION 1 Give a definition and an example of a default reasoning.

Solution Default reasoning is a reasoning in which it is allowed to draw plausible inferences from less-than-conclusive evidence in the absence of information to the contrary.

Example: Consider a statement *Birds fly*. Tweety, we are told, is a bird. From this, and the fact that birds fly, we conclude that Tweety can fly.

This conclusion, however is *defeasible*: Tweety may be an ostrich, a penguin, a bird with a broken wing, or a bird whose feet have been set in concrete. But as long as we don't have the evidence to the contrary (*Tweedy has a broken wing*) we accept the conclusion that *Tweedy can fly*.

QUESTION 2 Write the following natural language statement:

From the fact that it is not necessary that an elephant is not a bird we deduce that: it is not possible that an elephant is a bird or, if it is possible that an elephant is a bird, then it is not necessary that a bird flies.

as a formula

- (i) $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \mathbf{C}, \mathbf{I}, \cap, \cup, \Rightarrow\}}$,
- (ii) $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$.

Solution

- (i) We translate our statement into a formula

$A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \mathbf{C}, \mathbf{I}, \cap, \cup, \Rightarrow\}}$ as follows.

Propositional Variables: a, b .

a denotes statement: *an elephant is a bird*, b denotes a statement: *a bird flies*.

Propositional Modal Connectives: \mathbf{C}, \mathbf{I} .

\mathbf{C} denotes statement: *it is possible that*, \mathbf{I} denotes statement: *it is necessary that*.

Translation 1:

$$A_1 = (\neg \mathbf{I} \neg a \Rightarrow (\neg \mathbf{C} a \cup (\mathbf{C} a \Rightarrow \neg \mathbf{I} b))).$$

- (ii) **Now we translate** our statement into a formula

$A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$ as follows.

Propositional Variables: a, b, c .

a denotes statement: *it is necessary that an elephant is not a bird*,

b denotes statement: *it is possible that an elephant is a bird*,

c denotes a statement: *it is necessary that a bird flies.*

Translation 2:

$$A_2 = (\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))).$$

2. Main connective of the formula A_1 is: \Rightarrow , main connective of the formula A_2 is also \Rightarrow .
3. Degree of the formula A_1 is: 11, degree of the formula A_2 is: 6.
4. All proper, non-atomic sub-formulas of A_1 are:

$$\neg \mathbf{I}\neg a, (\neg \mathbf{C}a \cup (\mathbf{C}a \Rightarrow \neg \mathbf{I}b)), \mathbf{I}\neg a, \neg a, \neg \mathbf{C}a, (\mathbf{C}a \Rightarrow \neg \mathbf{I}b), \mathbf{C}a, \neg \mathbf{I}b, \mathbf{I}b$$

5. All non-atomic sub-formulas of A_2 are:

$$(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))), \neg a, (\neg b \cup (b \Rightarrow \neg c)), \neg b, (b \Rightarrow \neg c), \neg c$$

6. Find a model and a counter-model restricted to A_2 . Use short-hand notation. Show work.

A restricted model: $a = T, b = T, c = F$

Evaluation: $(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) = T$ for, for example $a = T$ and b, c any truth values. ($F \Rightarrow anything = T$).

$a = T$ gives 4 models (2^2 values on b and c .)

A Restricted counter-model: $a = F, b = T$ and $c = T$

Evaluation: $(\neg a \Rightarrow (\neg b \cup (b \Rightarrow \neg c))) = F$ if and only if

$\neg a = T$ and $(\neg b \cup (b \Rightarrow \neg c)) = F$, iff

$a = F, \neg b = F$ and $(b \Rightarrow \neg c) = F$, iff

$a = F, b = T$ and $(T \Rightarrow \neg c) = F$, iff

$a = F, b = T$ and $\neg c = F$ iff

$a = F, b = T$ and $c = T$

7. Statement: *There are more than 3 possible restricted counter-models of A_2 .* is not true. There is only one possible counter-model restricted to A_2 as shown by above evaluation.
8. Statement: *There are more than 2 possible restricted models of A_2 .* is true. There are 7 possible restricted models of A_2 . Justification: $2^3 - 1 = 7$.
9. List 3 models and 3 counter-models for A_2 by extending the model and the counter-model you have found in 5. to the VAR of all variables.

A model for A_2 is, by definition, any function

$$w : VAR \longrightarrow \{T, F\},$$

such that $w(A_2) = T$.

A restricted model for A_2 is, as defined in **7**. is a function

$$v : \{a, b, c\} \longrightarrow \{T, F\},$$

such that $v(A_2) = T$, i.e. for example:
 $A = T, b = T, c = F$.

We extend v to the set of all propositional variables VAR to obtain a (non restricted) model. Here are three of such extensions.

Model w_1

$$w_1 : VAR \longrightarrow \{T, F\}$$

$$w_1(a) = v(a) = T, \quad w_1(b) = v(b) = T, \quad w_1(c) = v(c) = F, \quad \text{and } w_1(x) = T, \quad \text{for all } x \in VAR - \{a, b, c\}.$$

Model w_2 :

$$w_2(a) = v(a) = T, \quad w_2(b) = v(b) = T, \quad w_2(c) = v(c) = F, \quad \text{and } w_2(x) = F, \\ \text{for all } x \in VAR - \{a, b, c\}.$$

Model w_3 :

$$w_3(a) = v(a) = T, \quad w_3(b) = v(b) = T, \quad w_3(c) = v(c) = F, \quad w_3(d) = F \text{ and } w_3(x) = T, \\ \text{for all } x \in VAR - \{a, b, c, d\}.$$

There is an many of such models, as extensions of v to the set VAR , i.e. as many as real numbers.

A counter-model for A_2 , by definition, is any function

$$w : VAR \longrightarrow \{T, F\},$$

such that $w(A_2) = F$.

A restricted counter-model for A_2 is, as defined in **6**. a function

$$v : \{a, b\} \longrightarrow \{T, F\},$$

such that $v(A) = F$, i.e. (only one) such that $v(a) = F, v(b) = T, v(c) = T$.

There is only one **restricted counter-model** v for A_2 .

We extend v to the set of all propositional variables VAR to obtain a (non restricted) counter-models. Here are three of such extensions.

Counter- model w_1 :

$$w_1(a) = v(a) = F, \quad w_1(b) = v(b) = T, \quad w_1(c) = v(c) = T, \quad \text{and } w_1(x) = F, \quad \text{for all } x \in VAR - \{a, b, c\}.$$

Counter- model w_2 :

$$w_2(a) = v(a) = T, \quad w_2(b) = v(b) = T, \quad w_2(c) = v(c) = T, \quad \text{and } w_2(x) = T, \\ \text{for all } x \in VAR - \{a, b, c\}.$$

There is an many of such counter- models, as extensions of v to the set VAR , i.e. as many as real numbers.

9. There are $2^{\aleph_0} = \mathcal{C}$ possible models for A_2 . There are $2^{\aleph_0} = \mathcal{C}$ possible counter-models for A_2 .

QUESTION 3

Show that (can't use TTables!)

$$\models ((\neg a \cup b) \Rightarrow (((c \cap d) \Rightarrow \neg d) \Rightarrow (\neg a \cup b)))$$

Solution:

Denote $A = (\neg a \cup b)$, and $B = ((c \cap d) \Rightarrow \neg d)$. Our formula becomes a substitution of a BASIC TAUTOLOGY ($A \Rightarrow (B \Rightarrow A)$) and hence is a tautology.

We define a 4 valued \mathbf{H}_4 logic semantics as follows. The language is $\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}$.

The logical connectives $\neg, \Rightarrow, \cup, \cap$ of \mathbf{H}_4 are operations in the set $\{F, \perp_1, \perp_2, T\}$, where $\{F < \perp_1 < \perp_2 < T\}$, defined as follows.

Conjunction \cap is a function $\cap: \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\}$, such that for any $a, b \in \{F, \perp_1, \perp_2, T\}$, $a \cap b = \min\{a, b\}$.

Disjunction \cup is a function $\cup: \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\}$, such that for any $a, b \in \{F, \perp_1, \perp_2, T\}$, $a \cup b = \max\{a, b\}$.

Implication \Rightarrow is a function $\Rightarrow: \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\}$, such that for any $a, b \in \{F, \perp_1, \perp_2, T\}$,

$$a \Rightarrow b = \begin{cases} T & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$

Negation:

$$\neg a = a \Rightarrow F.$$

QUESTION 4

Part 1 Write all Truth Tables for IMPLICATION and NEGATION in \mathbf{H}_4

Solution H_4 Implication

\Rightarrow	F	\perp_1	\perp_2	T
F	T	T	T	T
\perp_1	F	T	T	T
\perp_2	F	\perp_1	T	T
T	F	\perp_1	\perp_2	T

H_4 Negation

\neg	F	\perp_1	\perp_2	T
	T	F	F	F

Part 2 Verify whether

$$\models_{\mathbf{H}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

Solution Take any v such that $v(a) = \perp_1$ and $v(b) = \perp_2$.

$v * ((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = (\perp_1 \Rightarrow \perp_2) \Rightarrow (\neg \perp_1 \cup \perp_2) = T \Rightarrow (F \cup \perp_2) = T \Rightarrow \perp_2 = \perp_2$. This proves that our v is a counter-model and hence

$$\not\models_{\mathbf{H}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b)).$$

EXTRA CREDIT

1. Use Dedekind theorem to prove that the set R of real numbers is infinite.

Solution There are many 1 – 1 functions that map R onto its proper subset, here is one of the most obvious:

$$f(x) = 5^x.$$

2. Find a function f that is 1 – 1 and maps R ONTO $R - \{1, 8, 10\}$.

Solution There are many ways to construct the function f , but all constructions must follow the one presented (general case is presented in the proof of the Dedekind Theorem).

Idea: you must define 3 DISJOINT, 1 – 1 sequences a_n, b_n and c_n of real numbers and apply the definition of f from Challenge Problems, Question 3.

For example, let $n \geq 1$

$$a_n = \frac{1}{n}, \quad b_n = 1 + \frac{1}{n}, \quad c_n = 2 + \frac{1}{n}.$$

We define f as follows:

$$f(1) = a_1, \quad f(a_n) = a_{n+1}, \quad \text{all } n \geq 1,$$

$$f(8) = b_1, \quad f(b_n) = b_{n+1} \quad \text{all } n \geq 1,$$

$$f(10) = c_1, \quad f(c_n) = c_{n+1} \quad \text{all } n \geq 1,$$

and $f(x) = x$ for all other $x \in R$.