

**CSE371 Q2 PRACTICE Fall 2011**

**NAME**

**ID:**

**Math/CS**

**Solve all problems as an exercise.**

**QUESTION 1** Prove using proper logical equivalences (list them at each step) that

1.  $\neg(A \Leftrightarrow B) \equiv ((A \cap \neg B) \cup (\neg A \cap B)),$

**Solution :**

2.  $((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv ((B \Rightarrow C) \cup (A \Rightarrow B)).$

**Solution :**

**QUESTION 2** We define an EQUIVALENCE of LANGUAGES as follows:

Given two languages:

$\mathcal{L}_1 = \mathcal{L}_{CON_1}$  and  $\mathcal{L}_2 = \mathcal{L}_{CON_2}$ , for  $CON_1 \neq CON_2$ .

We say that they are **logically equivalent**, i.e.

$$\mathcal{L}_1 \equiv \mathcal{L}_2$$

if and only if the following conditions **C1**, **C2** hold.

**C1:** For every formula  $A$  of  $\mathcal{L}_1$ , there is a formula  $B$  of  $\mathcal{L}_2$ , such that

$$A \equiv B,$$

**C2:** For every formula  $C$  of  $\mathcal{L}_2$ , there is a formula  $D$  of  $\mathcal{L}_1$ , such that

$$C \equiv D.$$

**Prove that**  $\mathcal{L}_{\{\neg, \cap\}} \equiv \mathcal{L}_{\{\neg, \Rightarrow\}}$ .

**QUESTION 3** Given a proof system:

$$S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \mathcal{E} = \mathcal{F} \text{ AX} = \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}).$$

**Definition:** System  $S$  is sound if and only if

- (i) Axioms are tautologies and
- (ii) rules of inference are sound, i.e lead from all true premisses to a true conclusion.

1. Prove that  $S$  is *sound* under classical semantics.

2. Prove that  $S$  is *not sound* under  $\mathbf{K}$  semantics defined as follows.

**The language** is the same in case of classical logic.

**Connectives**  $\neg, \cup, \cap$  of  $\mathbf{K}$  are defined as in  $\mathbf{L}$  logic, i.e. for any  $a, b \in \{F, \perp, T\}$ ,

$$\neg \perp = \perp, \quad \neg F = T, \quad \neg T = F,$$

$$a \cup b = \max\{a, b\},$$

$$a \cap b = \min\{a, b\}.$$

**Implication** in Kleene's logic is defined as follows.

For any  $a, b \in \{F, \perp, T\}$ ,

$$a \Rightarrow b = \neg a \cup b.$$

3. Write a formal proof in  $S$  with 2 applications of the rule ( $r$ ).

**QUESTION 4** Prove, by constructing a formal proof that

$$\vdash_S ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B))).$$

**QUESTION 5**

$H$  is the following proof system:

$$H = ( \mathcal{L}_{\{\Rightarrow, \neg\}}, \mathcal{F}, AX = \{A1, A2, A3, A4\}, MP )$$

**A1**  $(A \Rightarrow (B \Rightarrow A)),$

**A2**  $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))),$

**A3**  $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

**A4**  $((A \Rightarrow B) \Rightarrow A)$

**MP** (Rule of inference)

$$(MP) \frac{A ; (A \Rightarrow B)}{B}$$

(1) Prove that  $H$  is SOUND under classical semantics.

(2) Does Deduction Theorem holds for  $H$ ? Justify shortly your answer.

(3) Is  $H$  COMPLETE with respect to all classical semantics tautologies? JUSTIFY your answer.

**QUESTION 6** Let  $H$  be the proof system defined in QUESTION 1.

(a) Prove the following:  $A \vdash_H (A \Rightarrow A)$

(b) We know that  $\vdash_H (\neg A \Rightarrow (A \Rightarrow B))$ . Prove, that  $\neg A, A \vdash_H B$ .

**QUESTION 7** Here are consecutive steps  $B_1, \dots, B_5$  in the formal proof in  $H_2$  of

$$(B \Rightarrow \neg\neg B)$$

$$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$$

$$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$$

$$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$$

$$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$$

$$B_5 = (B \Rightarrow \neg\neg B)$$

**Complete** the steps

$$B_1, \dots, B_5$$

of the proof by writing all details in the space provided below each step of the proof.

You have to write down **the proper substitutions and formulas** used at each step of the proof.

**You can use** the following already proved facts:

1.

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C),$$

2.

$$\vdash_{H_2} (\neg\neg B \Rightarrow B).$$