

## CSE371 Q2 PRACTICE SOLUTIONS

**Reminder:** We define **H** semantics operations  $\cup$  and  $\cap$  as follows

$$a \cup b = \max\{a, b\}, \quad a \cap b = \min\{a, b\}.$$

The **Truth Tables** for Implication and Negation are:

**H-Implication**

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	F	T	T
T	F	$\perp$	T

**H Negation**

$\neg$	F	$\perp$	T
	T	F	F

**QUESTION 1** We know that

$$v : VAR \longrightarrow \{F, \perp, T\}$$

is such that

$$v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp$$

under **H** semantics. **evaluate**  $v^*((((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b)))$ .

**Solution :**  $v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp$  under H semantics if and only if (we use shorthand notation)  $(a \cap b) = T$  and  $(a \Rightarrow c) = \perp$  if and only if  $a = T, b = T$  and  $(T \Rightarrow c) = \perp$  if and only if  $c = \perp$ . I.e. we have that

$$v^*((a \cap b) \Rightarrow (a \Rightarrow c)) = \perp \quad \text{iff} \quad a = T, b = T, c = \perp.$$

Now we can we **evaluate**  $v^*((((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b)))$  as follows (in shorthand notation).  
 $v^*((((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))) =$   
 $((((T \Rightarrow T) \Rightarrow (T \Rightarrow \neg \perp)) \cup (T \Rightarrow T))) =$   
 $((T \Rightarrow (T \Rightarrow F)) \cup T) = T.$

**We define** a 4 valued **L**<sub>4</sub> logic semantics as follows. The language is

$$\mathcal{L} = \mathcal{L}_{\{\neg, \Rightarrow, \cup, \cap\}}.$$

We define the logical connectives  $\neg, \Rightarrow, \cup, \cap$  of  $\mathbf{L}_4$  as the following operations in the set  $\{F, \perp_1, \perp_2, T\}$ , where  $\{F < \perp_1 < \perp_2 < T\}$ .

**Negation**  $\neg$

$$\neg: \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\},$$

such that

$$\neg \perp_1 = \perp_1, \quad \neg \perp_2 = \perp_2, \quad \neg F = T, \quad \neg T = F.$$

**Conjunction**  $\cap$

$$\cap: \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\}$$

such that for any  $a, b \in \{F, \perp_1, \perp_2, T\}$ ,

$$a \cap b = \min\{a, b\}.$$

**Disjunction**  $\cup$

$$\cup: \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\}$$

such that for any  $a, b \in \{F, \perp_1, \perp_2, T\}$ ,

$$a \cup b = \max\{a, b\}.$$

**Implication**  $\Rightarrow$

$$\Rightarrow: \{F, \perp_1, \perp_2, T\} \times \{F, \perp_1, \perp_2, T\} \longrightarrow \{F, \perp_1, \perp_2, T\},$$

such that for any  $a, b \in \{F, \perp_1, \perp_2, T\}$ ,

$$a \Rightarrow b = \begin{cases} \neg a \cup b & \text{if } a > b \\ T & \text{otherwise} \end{cases}$$

**QUESTION 2**

**Part 1** Write all TTables for  $\mathbf{L}_4$

**Solution :**

**$\mathbf{L}_4$  Negation**

$\neg$	F	$\perp_1$	$\perp_2$	T
	T	$\perp_1$	$\perp_2$	F

**$\mathbf{L}_4$  Disjunction**

$\cup$	F	$\perp_1$	$\perp_2$	T
F	F	$\perp_1$	$\perp_2$	T
$\perp_1$	$\perp_1$	$\perp_1$	$\perp_2$	T
$\perp_2$	$\perp_2$	$\perp_2$	$\perp_2$	T
T	T	T	T	T

**$\mathbf{L}_4$  Conjunction**

$\cap$	F	$\perp_1$	$\perp_2$	T
F	F	F	F	F
$\perp_1$	F	$\perp_1$	$\perp_1$	$\perp_1$
$\perp_2$	F	$\perp_1$	$\perp_2$	$\perp_2$
T	F	$\perp_1$	$\perp_2$	T

**$\mathbf{L}_4$ -Implication**

$\Rightarrow$	F	$\perp_1$	$\perp_2$	T
F	T	T	T	T
$\perp_1$	$\perp_1$	T	T	T
$\perp_2$	$\perp_2$	$\perp_2$	T	T
T	F	$\perp_1$	$\perp_2$	T

**Part 2** Verify whether

$$\models_{\mathbf{L}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b))$$

**Solution :** Let  $v$  be a truth assignment such that  $v(a) = v(b) = \perp_1$ .

We evaluate  $v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\perp_1 \Rightarrow \perp_1) \Rightarrow (\neg \perp_1 \cup \perp_1)) = (T \Rightarrow (\perp_1 \cup \perp_1)) = (T \Rightarrow \perp_1) = \perp_1$ .

This proves that  $v$  is a counter-model for our formula and

$$\not\models_{\mathbf{L}_4} ((a \Rightarrow b) \Rightarrow (\neg a \cup b)).$$

Observe that a  $v$  such that  $v(a) = v(b) = \perp_2$  is also a counter model, as  $v^*((a \Rightarrow b) \Rightarrow (\neg a \cup b)) = ((\perp_2 \Rightarrow \perp_2) \Rightarrow (\neg \perp_2 \cup \perp_2)) = (T \Rightarrow (\perp_2 \cup \perp_2)) = (T \Rightarrow \perp_2) = \perp_2$ .

**QUESTION 3** (5pts) Prove using proper logical equivalences (list them at each step) that

1.  $\neg(A \Leftrightarrow B) \equiv ((A \cap \neg B) \cup (\neg A \cap B)),$

**Solution:**  $\neg(A \Leftrightarrow B) \stackrel{def}{\equiv} \neg((A \Rightarrow B) \cap (B \Rightarrow A)) \stackrel{deMorgan}{\equiv} \neg(A \Rightarrow B) \cup \neg(B \Rightarrow A)$   
 $\stackrel{negimpl}{\equiv} ((A \cap \neg B) \cup (B \cap \neg A)) \stackrel{commut}{\equiv} ((A \cap \neg B) \cup (\neg A \cap B)).$

2.  $((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv ((B \Rightarrow C) \cup (A \Rightarrow B)).$

**Solution:**  $((B \cap \neg C) \Rightarrow (\neg A \cup B)) \stackrel{impl}{\equiv} \neg(B \cap \neg C) \cup (\neg A \cup B) \stackrel{deMorgan}{\equiv} (\neg B \cup \neg \neg C) \cup (\neg A \cup B)$   
 $\stackrel{dneg}{\equiv} (\neg B \cup C) \cup (\neg A \cup B) \stackrel{impl}{\equiv} ((B \Rightarrow C) \cup (A \Rightarrow B)).$

**QUESTION 4** We define an EQUIVALENCE of LANGUAGES as follows:

Given two languages:

$\mathcal{L}_1 = \mathcal{L}_{CON_1}$  and  $\mathcal{L}_2 = \mathcal{L}_{CON_2}$ , for  $CON_1 \neq CON_2$ .

We say that they are **logically equivalent**, i.e.

$$\mathcal{L}_1 \equiv \mathcal{L}_2$$

if and only if the following conditions **C1**, **C2** hold.

**C1:** For every formula  $A$  of  $\mathcal{L}_1$ , there is a formula  $B$  of  $\mathcal{L}_2$ , such that

$$A \equiv B,$$

**C2:** For every formula  $C$  of  $\mathcal{L}_2$ , there is a formula  $D$  of  $\mathcal{L}_1$ , such that

$$C \equiv D.$$

**Prove that**  $\mathcal{L}_{\{\neg, \cap\}} \equiv \mathcal{L}_{\{\neg, \Rightarrow\}}$ .

**Solution:** The equivalence of languages holds due to two definability of connectives equivalences:

$$(A \cap B) \equiv \neg(A \Rightarrow \neg B), \quad (A \Rightarrow B) \equiv \neg(A \cap \neg B).$$

**QUESTION 5** Given a proof system:

$$S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \mathcal{E} = \mathcal{F} \quad AX = \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, \quad (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}).$$

**Definition:** System  $S$  is sound if and only if

- (i) Axioms are tautologies and
- (ii) rules of inference are sound, i.e lead from all true premisses to a true conclusion.

1. Prove that  $S$  is *sound* under classical semantics.

**Solution:**

- (i) Both axioms of  $S$  are basic classical tautologies.
- (ii) Consider the rule of inference of  $S$ .

$$(r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}.$$

Assume that its premise (the only premise) is True, i.e. let  $v$  be any truth assignment, such that  $v^*(A \Rightarrow B) = T$ . We evaluate logical value of the conclusion under the truth assignment  $v$  as follows.

$$v^*(B \Rightarrow (A \Rightarrow B)) = v^*(B) \Rightarrow T = T$$

for any  $B$  and any value of  $v^*(B)$ .

2. Prove that  $S$  is *not sound* under  $\mathbf{K}$  semantics.

**Solution:** Axiom  $(A \Rightarrow A)$  is not a  $\mathbf{K}$  semantics tautology.

3. Write a formal proof in  $S$  with 2 applications of the rule  $(r)$ .

**Solution:** Required formal proof is a sequence  $A_1, A_2, A_3$ , where

$$A_1 = (A \Rightarrow A)$$

(Axiom)

$$A_2 = (A \Rightarrow (A \Rightarrow A))$$

Rule  $(r)$  application 1 for  $A = A, B = A$ .

$$A_3 = ((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A)))$$

Rule  $(r)$  application 2 for  $A = A, B = (A \Rightarrow A)$ .

**QUESTION 6** Prove, by constructing a formal proof that

$$\vdash_S ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B))).$$

**Solution:** Required formal proof is a sequence  $A_1, A_2$ , where

$$A_1 = (A \Rightarrow (\neg A \Rightarrow B))$$

Axiom

$$A_2 = ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))$$

Rule  $(r)$  application for  $A = A, B = (\neg A \Rightarrow B)$ .