

**CSE371 Q2 PRACTICE SOLUTIONS Fall 2011**

**QUESTION 1** Prove using proper logical equivalences (list them at each step) that

1.  $\neg(A \Leftrightarrow B) \equiv ((A \cap \neg B) \cup (\neg A \cap B)),$

**Solution:**  $\neg(A \Leftrightarrow B) \stackrel{def}{\equiv} \neg((A \Rightarrow B) \cap (B \Rightarrow A)) \stackrel{deMorgan}{\equiv} (\neg(A \Rightarrow B) \cup \neg(B \Rightarrow A))$   
 $\stackrel{negimpl}{\equiv} ((A \cap \neg B) \cup (B \cap \neg A)) \stackrel{commut}{\equiv} ((A \cap \neg B) \cup (\neg A \cap B)).$

2.  $((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv ((B \Rightarrow C) \cup (A \Rightarrow B)).$

**Solution:**  $((B \cap \neg C) \Rightarrow (\neg A \cup B)) \stackrel{impl}{\equiv} (\neg(B \cap \neg C) \cup (\neg A \cup B)) \stackrel{deMorgan}{\equiv} ((\neg B \cup \neg\neg C) \cup (\neg A \cup B))$   
 $\stackrel{dneg}{\equiv} ((\neg B \cup C) \cup (\neg A \cup B)) \stackrel{impl}{\equiv} ((B \Rightarrow C) \cup (A \Rightarrow B)).$

**QUESTION 2** We define an EQUIVALENCE of LANGUAGES as follows:

Given two languages:

$\mathcal{L}_1 = \mathcal{L}_{CON_1}$  and  $\mathcal{L}_2 = \mathcal{L}_{CON_2}$ , for  $CON_1 \neq CON_2$ .

We say that they are **logically equivalent**, i.e.

$$\mathcal{L}_1 \equiv \mathcal{L}_2$$

if and only if the following conditions **C1**, **C2** hold.

**C1:** For every formula  $A$  of  $\mathcal{L}_1$ , there is a formula  $B$  of  $\mathcal{L}_2$ , such that

$$A \equiv B,$$

**C2:** For every formula  $C$  of  $\mathcal{L}_2$ , there is a formula  $D$  of  $\mathcal{L}_1$ , such that

$$C \equiv D.$$

**Prove that**  $\mathcal{L}_{\{\neg, \cap\}} \equiv \mathcal{L}_{\{\neg, \Rightarrow\}}.$

**Solution:** The equivalence of languages holds due to two definability of connectives equivalences:

$$(A \cap B) \equiv \neg(A \Rightarrow \neg B), \quad (A \Rightarrow B) \equiv \neg(A \cap \neg B).$$

**QUESTION 3** Given a proof system:

$$S = (\mathcal{L}_{\{\neg, \Rightarrow\}}, \mathcal{E} = \mathcal{F} \quad AX = \{(A \Rightarrow A), (A \Rightarrow (\neg A \Rightarrow B))\}, \quad (r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}).$$

**Definition:** System  $S$  is sound if and only if

(i) Axioms are tautologies and

(ii) rules of inference are sound, i.e lead from all true premisses to a true conclusion.

1. Prove that  $S$  is *sound* under classical semantics.

**Solution:**

(i) Both axioms of  $S$  are basic classical tautologies.

(ii) Consider the rule of inference of  $S$ .

$$(r) \frac{(A \Rightarrow B)}{(B \Rightarrow (A \Rightarrow B))}.$$

Assume that its premise (the only premise) is True, i.e. let  $v$  be any truth assignment, such that  $v^*(A \Rightarrow B) = T$ . We evaluate logical value of the conclusion under the truth assignment  $v$  as follows.

$$v^*(B \Rightarrow (A \Rightarrow B)) = v^*(B) \Rightarrow T = T$$

for any  $B$  and any value of  $v^*(B)$ .

2. Prove that  $S$  is *not sound* under  $\mathbf{K}$  semantics.

**The language** of Kleene's logic is the same in case of classical logic.

**Connectives**  $\neg, \cup, \cap$  of  $\mathbf{K}$  are defined as in  $\mathbf{L}$  logic, i.e. for any  $a, b \in \{F, \perp, T\}$ ,

$$\neg \perp = \perp, \quad \neg F = T, \quad \neg T = F,$$

$$a \cup b = \max\{a, b\},$$

$$a \cap b = \min\{a, b\}.$$

**Implication** in Kleene's logic is defined as follows.

For any  $a, b \in \{F, \perp, T\}$ ,

$$a \Rightarrow b = \neg a \cup b.$$

2. Prove that  $S$  is *not sound* under  $\mathbf{K}$  semantics defined as follows.

**Solution:** Axiom  $(A \Rightarrow A)$  is not a  $\mathbf{K}$  semantics tautology. ANY truth assignment  $v$  such that  $v^*(A) = \perp$  is a counter-model.

3. Write a formal proof in  $S$  with 2 applications of the rule  $(r)$ .

**Solution:** Required formal proof is a sequence  $A_1, A_2, A_3$ , where

$$A_1 = (A \Rightarrow A)$$

(Axiom)

$$A_2 = (A \Rightarrow (A \Rightarrow A))$$

Rule  $(r)$  application 1 for  $A = A, B = A$ .

$$A_3 = ((A \Rightarrow A) \Rightarrow (A \Rightarrow (A \Rightarrow A)))$$

Rule  $(r)$  application 2 for  $A = A, B = (A \Rightarrow A)$ .

**QUESTION 4** Prove, by constructing a formal proof that

$$\vdash_S ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B))).$$

**Solution:** Required formal proof is a sequence  $A_1, A_2$ , where

$$A_1 = (A \Rightarrow (\neg A \Rightarrow B))$$

Axiom

$$A_2 = ((\neg A \Rightarrow B) \Rightarrow (A \Rightarrow (\neg A \Rightarrow B)))$$

Rule ( $r$ ) application for  $A = A, B = (\neg A \Rightarrow B)$ .

### QUESTION 5

$H$  is the following proof system:

$$H = ( \mathcal{L}_{\{\Rightarrow, \neg\}}, \mathcal{F}, AX = \{A1, A2, A3, A4\}, MP )$$

**A1**  $(A \Rightarrow (B \Rightarrow A))$ ,

**A2**  $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$ ,

**A3**  $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

**A4**  $((A \Rightarrow B) \Rightarrow A)$

**MP** (Rule of inference)

$$(MP) \frac{A ; (A \Rightarrow B)}{B}$$

(1) Prove that  $H$  is SOUND under classical semantics.

**Solution:** Soundness Theorem holds because all axioms of  $H$  are tautologies and MP leads from tautologies to a tautology.

(2) Why Deduction Theorem holds for  $H$ ?

**Solution:** System  $H$  extends by one extra axiom  $A3$  the proof system  $H_1$  for which we have proved the deduction theorem.

(3) Is  $H$  COMPLETE?

**Solution:** YES. Axioms  $A1, A2, A3$  of  $H$  are axioms of the system  $H_2$  from Chapter 8. It is stated in Chapter 8 and proved in Chapter 9 that Completeness Theorem holds for  $H_2$ .

**QUESTION 6** Let  $H$  be the proof system defined in QUESTION 1.

(a) Prove the following:  $A \vdash_H (A \Rightarrow A)$

**Solution 1:** Proof is as follows.

$$B_1 = (A \Rightarrow (A \Rightarrow A))$$

Axiom  $A1$  for  $B = A$

$$B_2 = A$$

Hypothesis

$$B_3 = (A \Rightarrow A)$$

$B_1, B_2$  and MP

**Solution 2:** We use Deduction Theorem.

$A \vdash_H (A \Rightarrow A)$  if and only if  $\vdash_H (A \Rightarrow (A \Rightarrow A))$ , what is true because  $(A \Rightarrow (A \Rightarrow A))$  is axiom A1.  
The proof is one element sequence:

$$B_1 = A \Rightarrow (A \Rightarrow A)$$

Axiom A1 for  $B = A$

(b) We know that  $\vdash_H (\neg A \Rightarrow (A \Rightarrow B))$ . Prove, that  $\neg A, A \vdash_H B$ .

**Solution 1:** We apply Deduction Theorem twice:

$\vdash_H (\neg A \Rightarrow (A \Rightarrow B))$  if and only if  $\neg A \vdash_H (A \Rightarrow B)$  if and only if  $\neg A, A \vdash_H B$ .

**Solution 2:** We construct the formal proof of  $\neg A, A \vdash_H B$  as follows.

$$B_1 = (\neg A \Rightarrow (A \Rightarrow B))$$

Assumption that  $\vdash_H (\neg A \Rightarrow (A \Rightarrow B))$

$$B_2 = \neg A$$

Hypothesis

$$B_3 = A$$

Hypothesis

$$B_4 = (A \Rightarrow B)$$

$B_1, B_2$  and MP

$$B_5 = B$$

$B_3, B_4$  and MP

**QUESTION 7** Here are consecutive steps  $B_1, \dots, B_5$  in the formal proof in  $H_2$  of

$$(B \Rightarrow \neg\neg B)$$

$$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$$

$$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$$

$$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$$

$$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$$

$$B_5 = (B \Rightarrow \neg\neg B)$$

**Complete** the steps

$$B_1, \dots, B_5$$

of the proof by writing all details in the space provided below each step of the proof.

You have to write down **the proper substitutions and formulas** used at each step of the proof.

**You can use** the following already proved facts:

1.

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C),$$

2.

$$\vdash_{H_2} (\neg\neg B \Rightarrow B).$$

**Solution** The completed comments are as follows.

$$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$$

Axiom A3 for  $A = B, B = \neg\neg B$

$$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$$

Already proved fact:  $\vdash_{H_2} (\neg\neg B \Rightarrow B)$  for  $B = \neg B$

$$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$$

$B_1, B_2$  and MP, i.e.

$$\frac{(\neg\neg\neg B \Rightarrow \neg B); ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))}{((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)}$$

$$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$$

Axiom A1 for  $A = B, B = \neg\neg\neg B$

$$B_5 = (B \Rightarrow \neg\neg B)$$

$B_3, B_4$  and already proved fact:  
 $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$  for  
 $A = B, B = (\neg\neg\neg B \Rightarrow B), C = \neg\neg B$  i.e.

$$(B \Rightarrow (\neg\neg\neg B \Rightarrow B)), ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B) \vdash_{H_2} (B \Rightarrow \neg\neg B)$$