

**CSE371 Q2 SOLUTIONS Fall 2009**

**QUESTION 1** Use the fact that  $v : VAR \rightarrow \{F, \perp, T\}$  be such that  $v^*((a \cap b) \Rightarrow \neg b) = \perp$  under **L** semantics to evaluate  $v^*((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b)$ . Use shorthand notation.

**L Negation**

$\neg$	F	$\perp$	T
	T	$\perp$	F

**L Disjunction**

$\cup$	F	$\perp$	T
F	F	$\perp$	T
$\perp$	$\perp$	$\perp$	T
T	T	T	T

**L Conjunction**

$\cap$	F	$\perp$	T
F	F	F	F
$\perp$	F	$\perp$	$\perp$
T	F	$\perp$	T

**L-Implication**

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	$\perp$	T	T
T	F	$\perp$	T

**Solution :**  $((a \cap b) \Rightarrow \neg b) = \perp$  in two cases.

**C1**  $(a \cap b) = \perp$  and  $\neg b = F$ .

**C2**  $(a \cap b) = T$  and  $\neg b = \perp$ .

**Case C1:**  $\neg b = F$ , i.e.  $b = T$ , and hence  $(a \cap T) = \perp$  iff  $a = \perp$ . We get that  $v$  is such that  $v(a) = \perp$  and  $v(b) = T$ .

**We evaluate:**  $v^*((b \Rightarrow \neg a) \Rightarrow (a \Rightarrow \neg b)) \cup (a \Rightarrow b) = (((T \Rightarrow \neg \perp) \Rightarrow (\perp \Rightarrow \neg T)) \cup (\perp \Rightarrow T)) = ((\perp \Rightarrow \perp) \cup T) = T$ .

**Case C2:**  $\neg b = \perp$ , i.e.  $b = \perp$ , and hence  $(a \cap \perp) = T$  what is impossible, hence  $v$  from case C1 is the only one.

**QUESTION 2** Prove using proper logical equivalences (list them at each step) that

$$\neg((A \Rightarrow \neg B) \cup (B \Rightarrow \neg A)) \equiv (A \cap B).$$

**Solution :**  $\neg((A \Rightarrow \neg B) \cup (B \Rightarrow \neg A)) \equiv^{deMorg} (\neg(A \Rightarrow \neg B) \cap \neg(B \Rightarrow \neg A)) \equiv^{negimpl} ((A \cap \neg\neg B) \cap (B \cap \neg\neg A)) \equiv^{dneg} ((A \cap B) \cap (B \cap A)) \equiv^{assoc,comm} (A \cap B)$ .

**QUESTION 3** Given a proof system:

$$S = (\mathcal{L}_{\{\cup, \Rightarrow\}}, \mathcal{E} = \mathcal{F} \quad AX = \{A1, A2\}, \mathcal{R} = \{(r)\} ),$$

where

$$A1 = (A \Rightarrow (A \cup B)), \quad A2 = (A \Rightarrow (B \Rightarrow A))$$

and

$$(r) \frac{(A \Rightarrow B)}{(A \Rightarrow (A \Rightarrow B))}$$

**1. Solution:** Prove that  $S$  is *sound* under classical semantics.

**Solution:** Axioms of  $S$  are basic classical tautologies. The proof of soundness of the rule of inference is the following.

Assume  $(A \Rightarrow B) = T$ . Hence the logical value of conclusion is  $(A \Rightarrow (A \Rightarrow B)) = (A \Rightarrow T) = T$  for all  $A$ .

**2.** Determine whether  $S$  is *sound* or *not sound* under  $\mathbf{K}$  semantics.

**K semantics** differ from Lukasiewicz's semantics only in a case on implication only. This table is:

**K-Implication**

$\Rightarrow$	F	$\perp$	T
F	T	T	T
$\perp$	$\perp$	$\perp$	T
T	F	$\perp$	T

**Solution 1:**  $S$  is not sound under  $\mathbf{K}$  semantics. Let's take truth assignment such that  $A = \perp, B = \perp$ . The logical value of axiom A1 is as follows.

$$(A \Rightarrow (A \cup B)) = (\perp \Rightarrow (\perp \cup \perp)) = \perp \text{ and } \not\models_{\mathbf{K}} (A \Rightarrow (A \cup B)).$$

**Observe** that the  $v$  such that  $A = \perp, B = \perp$  is not the only  $v$  that makes  $A1 \neq T$ , i.e. proves that  $\not\models_{\mathbf{K}} A1$ .

$(A \Rightarrow (A \cup B)) \neq T$  if and only if  $(A \Rightarrow (A \cup B)) = F$  or  $(A \Rightarrow (A \cup B)) = \perp$ . The first case is impossible because A1 is a classical tautology.

Consider the second case.  $(A \Rightarrow (A \cup B)) = \perp$  in two cases.

**c1**  $A = \perp$  and  $(A \cup B) = F$ , i.e.  $(\perp \cup B) = F$ , what is impossible.

**c2**  $A = T$  and  $(A \cup B) = \perp$ , i.e.  $(T \cup B) = \perp$ , what is impossible.

**c3**  $A = \perp$  and  $(A \cup B) = \perp$ , i.e.  $(\perp \cup B) = \perp$ . This is possible for  $B = \perp$  or  $B = F$ , i.e. when  $A = \perp, B = \perp$  or  $A = \perp, B = F$ .

From the above Observation we get second solution.

**Solution 2:**  $S$  is not sound under  $\mathbf{K}$  semantics. Axiom A1 is not  $\mathbf{K}$  semantics tautology. There are exactly two truth assignments  $v$ , such that  $v \not\models A1$ . One is, as defined in Solution 1:  $A = \perp, B = \perp$ . The second is  $A = \perp, B = F$ .

#### QUESTION 4

**1.** Write a formal proof  $A_1, A_2, A_3$  in  $S$  from the QUESTION 3 with 2 applications of the rule  $(r)$  that starts with axiom A1, i.e such that  $A_1 = A1$ .

**Solution:** The formal proof  $A_1, A_2, A_3$  is as follows.

$$A_1 = (A \Rightarrow (A \cup B))$$

Axiom

$$A_2 = (A \Rightarrow (A \Rightarrow (A \cup B)))$$

Rule  $(r)$  application for  $A = A$  and  $B = (A \cup B)$

$$A_3 = (A \Rightarrow (A \Rightarrow (A \Rightarrow (A \cup B))))$$

Rule  $(r)$  application for  $A = A$  and  $B = (A \Rightarrow (A \cup B))$ .

2. Use results from QUESTION 3 to determine whether  $\models_{\mathbf{K}} A_3$ .

**Solution 1:** We use the two  $v$  from QUESTION 3 to evaluate the logical value of  $A_3$ . Namely we evaluate:  
 $v^*(A \Rightarrow (A \Rightarrow (A \Rightarrow (A \cup B)))) = (\perp \Rightarrow (\perp \Rightarrow (\perp \Rightarrow (\perp \cup \perp)))) = \perp$ , or  $v^*(A \Rightarrow (A \Rightarrow (A \Rightarrow (A \cup B)))) = (\perp \Rightarrow (\perp \Rightarrow (\perp \Rightarrow (\perp \cup F)))) = \perp$ . Both cases prove that  $\not\models_{\mathbf{K}} A_3$ .

**Solution 2:** We know that  $S$  is not sound, because there is  $v$  for which  $A_1 = A_1 = \perp$ , as evaluated in Question 3. We prove that the rule  $(r)$  preserves the logical value  $\perp$  under any  $v$  such that  $A_1 = \perp$ . as follows.

Let the premiss  $(A \Rightarrow B) = \perp$ , the logical value of the conclusion is hence  $(A \Rightarrow \perp) = \perp$  for  $A = \perp, T$  and  $(A \Rightarrow \perp) = T$  for  $A = F$ .

The case  $A = F$  evaluates the premiss  $(A \Rightarrow B) = (F \Rightarrow B) = T$  for all  $B$ , what contradicts the assumption that  $(A \Rightarrow B) = \perp$ . We must hence have  $A = \perp$ . But all possible  $v$  for which  $A_1 = \perp$  are such that  $A = \perp$ , what end the proof.

It means that any  $A$  such that  $A$  has proof that starts with axiom  $A_1$  and then multiple applications of the rule  $(r)$  is evaluated to  $\perp$  under all  $v$ , such that  $v^*(A_1) = \perp$ . Hence, in particular,  $\not\models_{\mathbf{K}} A_3$ .

3. Write a formal proof  $A_1, A_2$  in  $S$  from the QUESTION 3 with 1 application of the rule  $(r)$  that starts with axiom  $A_2$ , i.e such that  $A_1 = A_2$ .

**Solution:** The formal proof  $A_1, A_2$  is as follows.

$$A_2 = (A \Rightarrow (B \Rightarrow A))$$

Axiom

$$A_1 = (A \Rightarrow (A \Rightarrow (B \Rightarrow A)))$$

Rule  $(r)$  application for  $A = A$  and  $B = (B \Rightarrow A)$ .

4. Use results from QUESTION 3 to determine whether  $\models A_2$ .

**Solution:** System  $S$  is sound under classical semantics, hence by the soundness theorem we get that

$$\models (A \Rightarrow (A \Rightarrow (B \Rightarrow A))),$$

as it has a proof in  $S$ .

**QUESTION 5** Prove, by constructing a formal proof in  $S$  from the QUESTION 3 that

$$\vdash_S (A \Rightarrow (A \Rightarrow (A \Rightarrow (A \Rightarrow A)))).$$

**Solution:**  $A_2 = (A \Rightarrow (A \Rightarrow A))$

Axiom for  $B = A$

$$A_2 = (A \Rightarrow (A \Rightarrow (A \Rightarrow A)))$$

Rule  $(r)$  application for  $A = A$  and  $B = (A \Rightarrow A)$ .

$$(A \Rightarrow (A \Rightarrow (A \Rightarrow (A \Rightarrow A))))$$

Rule  $(r)$  application for  $A = A$  and  $B = (A \Rightarrow (A \Rightarrow A))$ .