

CSE371 Q2 SOLUTIONS Fall 2011

QUESTION 1

1. Given a formula $A = ((B \cap \neg C) \Rightarrow (\neg A \cup B))$ of a language $\mathcal{L}_1 = \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$. FIND a formula B of a language $\mathcal{L}_2 = \mathcal{L}_{\{\neg, \Rightarrow\}}$, such that $A \equiv B$.
LIST all proper logical equivalences used at each step.

Solution :

$$A = ((B \cap \neg C) \Rightarrow (\neg A \cup B)) \equiv ((B \cap \neg C) \Rightarrow (A \Rightarrow B)) \equiv (\neg(B \Rightarrow \neg\neg C) \Rightarrow (A \Rightarrow B)) \equiv (\neg(B \Rightarrow C) \Rightarrow (A \Rightarrow B)) = B$$

Equivalences used:

1. $(\neg A \cup B) \equiv (A \Rightarrow B)$
2. $(A \cap B) \equiv \neg(A \Rightarrow \neg B)$
3. $\neg\neg A \equiv A$
2. Prove that $\mathcal{L}_1 \equiv \mathcal{L}_2$.

We define the EQUIVALENCE of LANGUAGES as follows:

Given two languages:

$\mathcal{L}_1 = \mathcal{L}_{CON_1}$ and $\mathcal{L}_2 = \mathcal{L}_{CON_2}$, for $CON_1 \neq CON_2$.

We say that they are **logically equivalent**, i.e.

$$\mathcal{L}_1 \equiv \mathcal{L}_2$$

if and only if the following conditions **C1**, **C2** hold.

C1: For every formula A of \mathcal{L}_1 , there is a formula B of \mathcal{L}_2 , such that

$$A \equiv B,$$

C2: For every formula C of \mathcal{L}_2 , there is a formula D of \mathcal{L}_1 , such that

$$C \equiv D.$$

Solution : We have to prove that

$$\mathcal{L}_{\{\neg, \Rightarrow\}} \equiv \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}.$$

Condition **C1** holds because $\{\neg, \Rightarrow\} \subseteq \{\neg, \cap, \cup, \Rightarrow\}$.

Condition **C1** holds because

$$(A \cap B) \equiv \neg(A \Rightarrow \neg B) \text{ and } (A \cup B) \equiv (\neg A \Rightarrow B).$$

QUESTION 2

S is the following proof system:

$$S = (\mathcal{L}_{\{\Rightarrow, \neg\}}, \mathcal{F}, AX = \{A1, A2, A3, A4\}, MP)$$

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

A4 $((A \Rightarrow B) \Rightarrow B) \Rightarrow A$

MP (Rule of inference)

$$(MP) \frac{A ; (A \Rightarrow B)}{B}$$

(1) Does Deduction Theorem holds for S ? Justify shortly your answer.

Solution : Yes, it does as only axioms A1 and A2 were used in its proof.

(2) Is S COMPLETE with respect to classical semantics? JUSTIFY your answer.

Solution : NO, is NOT Complete as it is not SOUND. Axiom A4 is not a tautology.

QUESTION 3

Let S be from QUESTION 2.

The following Lemma holds in the system S .

LEMMA For any $A, B, C \in \mathcal{F}$,

(a) $(A \Rightarrow B), (B \Rightarrow C) \vdash_S (A \Rightarrow C)$,

(b) $(A \Rightarrow (B \Rightarrow C)) \vdash_S (B \Rightarrow (A \Rightarrow C))$.

Complete the proof sequence (in S)

$$B_1, \dots, B_9$$

by providing comments how each step of the proof was obtained.

Solution

$B_1 = (A \Rightarrow B)$
Hypothesis

$B_2 = (\neg\neg A \Rightarrow A)$
Already Proven

$B_3 = (\neg\neg A \Rightarrow B)$
Lemma a for $A = \neg\neg A, B = A, C = B$, in B_1, B_2 i.e.

$$(\neg\neg A \Rightarrow A), (A \Rightarrow B) \vdash (\neg\neg A \Rightarrow B)$$

$$B_4 = (B \Rightarrow \neg\neg B)$$

Formula 5

$$B_5 = (\neg\neg A \Rightarrow \neg\neg B)$$

Lemma **a** on B_3, B_4 for $A = \neg\neg A, B = B, C = \neg\neg B$

$$(\neg\neg A \Rightarrow B), (B \Rightarrow \neg\neg B) \vdash (\neg\neg A \Rightarrow \neg\neg B)$$

$$B_6 = ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$$

ALREADY PROVED

$$B_7 = (\neg B \Rightarrow \neg A)$$

B_5, B_6 and MP on B_5, B_6

$$\frac{(\neg\neg A \Rightarrow \neg\neg B); ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))}{(\neg B \Rightarrow \neg A)}$$

$$B_8 = (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$$

$B_1 - B_7$

$$B_9 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$$

Deduction Theorem on B_8

Extra Credit Prove the above LEMMA (b), i.e prove that

$$(b) \quad (A \Rightarrow (B \Rightarrow C)) \vdash_S (B \Rightarrow (A \Rightarrow C)).$$

Solution : Deduction Theorem Holds for S . By Deduction Theorem applied twice we have that $(A \Rightarrow (B \Rightarrow C)) \vdash_S (B \Rightarrow (A \Rightarrow C))$ iff

$$(A \Rightarrow (B \Rightarrow C)), B \vdash_S (A \Rightarrow C) \text{ iff}$$

$$(A \Rightarrow (B \Rightarrow C)), B, A \vdash_S C$$

The proof of C from $(A \Rightarrow (B \Rightarrow C)), B, A$ is the following.

$$B_1 = (A \Rightarrow (B \Rightarrow C))$$

Hypothesis

$$B_2 = A$$

Hypothesis

$$B_3 = (B \Rightarrow C)$$

B_1, B_2 and MP

$$B_4 = B$$

Hypothesis

$$B_5 = C$$

B_3, B_4 and MP