

CSE371 Q3 PRACTICE SOLUTIONS Fall 2009

**QUESTION 1**

$H$  is the following proof system:

$$H = ( \mathcal{L}_{\{\Rightarrow, \neg\}}, \mathcal{F}, AX = \{A1, A2, A3\}, MP )$$

**A1**  $(A \Rightarrow (B \Rightarrow A))$ ,

**A2**  $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$ ,

**A3**  $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

**A4**  $((A \Rightarrow B) \Rightarrow A)$

**MP** (Rule of inference)

$$(MP) \frac{A ; (A \Rightarrow B)}{B}$$

(1) Prove that  $H$  is SOUND under classical semantics.

**Solution:** Soundness Theorem holds because all axioms of  $H$  are tautologies and MP leads from tautologies to a tautology.

(2) Why Deduction Theorem holds for  $H$ ?

**Solution:** System  $H$  extends by one extra axiom  $A3$  the proof system  $H_1$  for which we have proved the deduction theorem.

(3) Is  $H$  COMPLETE?

**Solution:** YES. Axioms  $A1, A2, A3$  of  $H$  are axioms of the system  $H_2$  from Chapter 8. It is stated in Chapter 8 and proved in Chapter 9 that Completeness Theorem holds for  $H_2$ .

**QUESTION 2**  $S$  is the following (sound) proof system:

$$S = ( \mathcal{L}_{\{\Rightarrow, \cap\}}, \mathcal{F}, AX = \{A1\} \mathcal{R} = \{(r_1), (r_2)\} ),$$

where

**Axiom:**  $A1 = (B \Rightarrow (A \Rightarrow B))$ ,

**Rules:**

$$(r_1) \frac{A ; B}{(A \cap B)}$$

$$(r_2) \frac{A ; (C \cap D)}{(A \Rightarrow (C \cap D))}$$

For the sequence  $B_1, B_2, B_3, B_4$  of formulas of  $\mathcal{L}_{\{\Rightarrow, \cap\}}$  defined below determine if  $B_1, B_2, B_3, B_4$  form a FORMAL PROOF in  $S$ .

If YES, provide comments how each step of the proof was obtained. Write your comments in the space between the steps.

If NOT, write the reason in a proper space between the steps.

**Solution:**

$$B_1 = (A \Rightarrow (B \Rightarrow A)),$$

Axiom A1

$$B_2 = (B \Rightarrow (A \Rightarrow B)),$$

Axiom A1 for  $A = B, B = A$

$$B_3 = ((B \Rightarrow (A \Rightarrow B)) \cap (A \Rightarrow (B \Rightarrow A))),$$

$B_1, B_2$  and  $(r_1)$  for  $A = B_2, B = B_1$

$$B_4 = ((A \Rightarrow (B \Rightarrow A)) \Rightarrow ((B \Rightarrow (A \Rightarrow B)) \cap (A \Rightarrow (B \Rightarrow A))))$$

$B_1, B_3$  and  $(r_2)$  for  $A = B_2, (C \cap D) = B_3,$   
i.e  $C = (B \Rightarrow (A \Rightarrow B)), D = (A \Rightarrow (B \Rightarrow A)).$

**QUESTION 3** Let  $H$  be the proof system defined in QUESTION 1.

(a) Prove the following:  $A \vdash_H (A \Rightarrow A)$

**Solution 1:** Proof is as follows.

$$B_1 = (A \Rightarrow (A \Rightarrow A))$$

Axiom A1 for  $B = A$

$$B_2 = A$$

Hypothesis

$$B_3 = (A \Rightarrow A)$$

$B_1, B_2$  and MP

**Solution 2:** We use Deduction Theorem.

$A \vdash_H (A \Rightarrow A)$  if and only if  $\vdash_H (A \Rightarrow (A \Rightarrow A))$ , what is true because  $(A \Rightarrow (A \Rightarrow A))$  is axiom A1.  
The proof is one element sequence:

$$B_1 = A \Rightarrow (A \Rightarrow A)$$

Axiom A1 for  $B = A$

(b) We know that  $\vdash_H (\neg A \Rightarrow (A \Rightarrow B))$ . Prove, that  $\neg A, A \vdash_H B$ .

**Solution 1:** We apply Deduction Theorem twice:

$\vdash_H (\neg A \Rightarrow (A \Rightarrow B))$  if and only if  $\neg A \vdash_H (A \Rightarrow B)$  if and only if  $\neg A, A \vdash_H B$ .

**Solution 2:** We construct the formal proof of  $\neg A, A \vdash_H B$  as follows.

$$B_1 = (\neg A \Rightarrow (A \Rightarrow B))$$

Assumption that  $\vdash_H (\neg A \Rightarrow (A \Rightarrow B))$

$$B_2 = \neg A$$

Hypothesis

$$B_3 = A$$

Hypothesis

$$B_4 = (A \Rightarrow B)$$

$B_1, B_2$  and MP

$$B_5 = B$$

$B_3, B_4$  and MP

**QUESTION 4** Here are consecutive steps  $B_1, \dots, B_5$  in a proof of  $(B \Rightarrow \neg\neg B)$  in  $H_2$ .

The comments included are incomplete.

**Complete the comments** by writing all details in the space provided below each step of the proof. You have to write down **the proper substitutions and formulas** used at each step of the proof.

$$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$$

Axiom A3

$$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$$

Already proved fact:  $\vdash_{H_2}(\neg\neg B \Rightarrow B)$

$$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$$

(MP)

$$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$$

Axiom A1

$$B_5 = (B \Rightarrow \neg\neg B)$$

Already proved fact:  $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$

**Solution** The completed comments are as follows.

$$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$$

Axiom A3 for  $A = B, B = \neg\neg B$

$$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$$

Already proved fact:  $\vdash_{H_2}(\neg\neg B \Rightarrow B)$  for  $B = \neg B$

$$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$$

$B_1, B_2$  and MP, i.e.

$$\frac{(\neg\neg\neg B \Rightarrow \neg B); ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))}{((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)}$$

$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$   
Axiom A1 for  $A = B, B = \neg\neg\neg B$

$B_5 = (B \Rightarrow \neg\neg B)$   
 $B_3, B_4$  and already proved fact:  
 $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$  for  
 $A = B, B = (\neg\neg\neg B \Rightarrow B), C = \neg\neg B$  i.e.

$(B \Rightarrow (\neg\neg\neg B \Rightarrow B)), ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B) \vdash_{H_2} (B \Rightarrow \neg\neg B)$