

CSE371 Q3 PRACTICE SOLUTIONS Fall 2010

QUESTION 1

H is the following proof system:

$$H = (\mathcal{L}_{\{\Rightarrow, \neg\}}, \mathcal{F}, AX = \{A1, A2, A3\}, MP)$$

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

A4 $((A \Rightarrow B) \Rightarrow A) \Rightarrow A$

MP (Rule of inference)

$$(MP) \frac{A ; (A \Rightarrow B)}{B}$$

(1) Prove that H is SOUND under classical semantics.

Solution: Soundness Theorem holds because all axioms of H are tautologies and MP leads from tautologies to a tautology.

(2) Why Deduction Theorem holds for H ?

Solution: System H extends by one extra axiom $A3$ the proof system H_1 for which we have proved the deduction theorem.

(3) Is H COMPLETE?

Solution: YES. Axioms A1, A2, A3 of H are axioms of the system H_2 from Chapter 8. It is stated in Chapter 8 and proved in Chapter 9 that Completeness Theorem holds for H_2 .

QUESTION 2 Let H be the proof system defined in QUESTION 1.

(a) Prove the following: $A \vdash_H (A \Rightarrow A)$

Solution 1: Proof is as follows.

$$B_1 = (A \Rightarrow (A \Rightarrow A))$$

Axiom A1 for $B = A$

$$B_2 = A$$

Hypothesis

$$B_3 = (A \Rightarrow A)$$

B_1, B_2 and MP

Solution 2: We use Deduction Theorem.

$A \vdash_H (A \Rightarrow A)$ if and only if $\vdash_H (A \Rightarrow (A \Rightarrow A))$, what is true because $(A \Rightarrow (A \Rightarrow A))$ is axiom A1.
The proof is one element sequence:

$$B_1 = A \Rightarrow (A \Rightarrow A) \\ \text{Axiom A1 for } B = A$$

(b) We know that $\vdash_H (\neg A \Rightarrow (A \Rightarrow B))$. Prove, that $\neg A, A \vdash_H B$.

Solution 1: We apply Deduction Theorem twice:

$\vdash_H (\neg A \Rightarrow (A \Rightarrow B))$ if and only if $\neg A \vdash_H (A \Rightarrow B)$ if and only if $\neg A, A \vdash_H B$.

Solution 2: We construct the formal proof of $\neg A, A \vdash_H B$ as follows.

$$B_1 = (\neg A \Rightarrow (A \Rightarrow B)) \\ \text{Assumption that } \vdash_H (\neg A \Rightarrow (A \Rightarrow B))$$

$$B_2 = \neg A \\ \text{Hypothesis}$$

$$B_3 = A \\ \text{Hypothesis}$$

$$B_4 = (A \Rightarrow B) \\ B_1, B_2 \text{ and MP}$$

$$B_5 = B \\ B_3, B_4 \text{ and MP}$$

QUESTION 4 Here are consecutive steps B_1, \dots, B_5 in a proof of $(B \Rightarrow \neg\neg B)$ in H_2 .

Complete the steps

$$B_1, \dots, B_5$$

of the proof by writing all details in the space provided below each step of the proof.

You have to write down **the proper substitutions and formulas** used at each step of the proof.

$$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)) \\ \text{Axiom A3}$$

$$B_2 = (\neg\neg\neg B \Rightarrow \neg B) \\ \text{Already proved fact: } \vdash_{H_2} (\neg\neg B \Rightarrow B)$$

$$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B) \\ \text{(MP)}$$

$$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B)) \\ \text{Axiom A1}$$

$$B_5 = (B \Rightarrow \neg\neg B) \\ \text{Already proved fact: } (A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$$

Solution The completed comments are as follows.

$$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$$

Axiom A3 for $A = B, B = \neg\neg B$

$$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$$

Already proved fact: $\vdash_{H_2}(\neg\neg B \Rightarrow B)$ for $B = \neg B$

$$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$$

B_1, B_2 and MP, i.e.

$$\frac{(\neg\neg\neg B \Rightarrow \neg B); ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))}{((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)}$$

$$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$$

Axiom A1 for $A = B, B = \neg\neg\neg B$

$$B_5 = (B \Rightarrow \neg\neg B)$$

B_3, B_4 and already proved fact:
 $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$ for
 $A = B, B = (\neg\neg\neg B \Rightarrow B), C = \neg\neg B$ i.e.

$$(B \Rightarrow (\neg\neg\neg B \Rightarrow B)), ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B) \vdash_{H_2} (B \Rightarrow \neg\neg B)$$

QUESTION 4 Let A be a formula

$$((\neg a \Rightarrow \neg b) \Rightarrow (c \cup a))$$

and let v be such that

$$v(a) = T, \quad v(b) = F, \quad v(c) = F.$$

Evaluate A', B_1, \dots, B_n as defined by the above definition. Write carefully all steps. You can use shorthand notation.

Evaluate A', B_1, \dots, B_n as defined by the following definition.

Definition Let A be a formula and b_1, b_2, \dots, b_n be all propositional variables that occur in A .

Let v be variable assignment $v : VAR \longrightarrow \{T, F\}$.

We define, for A, b_1, b_2, \dots, b_n and v a corresponding formulas A', B_1, B_2, \dots, B_n as follows: (for $i = 1, 2, \dots, n$.)

$$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases}$$

$$B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$$

for $i = 1, 2, \dots, n$.

Solution We use full notation.

In this case $n = 3$ and $b_1 = a, b_2 = b, b_3 = c$, and $v^*(A) = v^*((\neg a \Rightarrow \neg b) \Rightarrow c) = ((\neg v(a) \Rightarrow \neg v(b)) \Rightarrow v(c \cup a)) = ((\neg T \Rightarrow \neg F) \Rightarrow (F \cup T)) = (T \Rightarrow T) = T$.

The corresponding A', B_1, B_2, B_3 are:

$$A' = A = ((\neg a \Rightarrow \neg b) \Rightarrow c) \quad (\text{as } v^*(A) = T),$$

$$B_1 = a \quad (\text{as } v(a) = T),$$

$$B_2 = \neg b \quad (\text{as } v(b) = F).$$

$$B_3 = \neg c \quad (\text{as } v(c) = F).$$

QUESTION 5 Consider a system **RS2** obtained from **RS** by changing the sequence Γ' into Γ and Δ into Δ' in all of the rules of inference of **RS**.

1. Construct a decomposition tree of A from the QUESTION 2 in **RS2**.

Solution construction is similar to **RS**, except that now we traverse sequences from RIGHT to left.

2. Define in your own words, for any A , the decomposition tree T_A in **RS2**.

Solution the definition is similar as in the book.