CSE371 Q3 SOLUTIONS Fall 2009

QUESTION 1

H is the following proof system:

$$H = (\ \mathcal{L}_{\{\Rightarrow\}},\ \mathcal{F},\ AX = \{A1, A2\},\ \ MP\ )$$

A1 $(B \Rightarrow (A \Rightarrow B))$,

A2 $((B \Rightarrow (A \Rightarrow C)) \Rightarrow ((B \Rightarrow A) \Rightarrow (B \Rightarrow C)))$,

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

(1) Prove that H is SOUND under classical semantics.

Solution: $A1, A2$ are substitutions of axioms $A1, A2$ the sound system H_1 of our Book, and MP rule is sound.

(2) Does Deduction Theorem holds for H? Justify your answer.

Solution: Yes, it does. Our system H contains all axioms of H_1 (our $A1, A2$ are substitutions of the axioms of H_1) and these were the only axioms needed for the proof of Deduction Theorem.

(3) Is H COMPLETE with respect to all classical semantics tautologies? Justify your answer.

Solution: The system H **is not complete** under classical semantics. It means that not all classical tautologies have a proof in H. We have proved that one needs negation and one of other connectives $\cup, \cap, \Rightarrow$ to express all classical connectives, and hence all classical tautologies. Our language contains only implication and one can't express negation in terms of implication and hence we can't provide a proof of any tautology i.e. its logically equivalent form in our language.

QUESTION 2 (10pts) Here are consecutive steps $B_1, ..., B_9$ in a proof of

$$((B \Rightarrow A) \Rightarrow (\neg A \Rightarrow \neg B))$$

in H_2. The comments included are incomplete.

Complete the comments by writing all details in the space provided below each step of the proof. You have to write down the proper substitutions and formulas used at each step of the proof.

$B_1 =\ (B \Rightarrow A)$
Hypothesis

$B_2 =\ (\neg\neg B \Rightarrow B)$
Already proved formula: $(\neg\neg A \Rightarrow A)$ for $A = B$

$B_3 = \ (\neg\neg B \Rightarrow A)$
B_1, B_2 and already proved fact: $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$ for $A = \neg\neg B, B = B, C = A$ i.e.

$$(\neg\neg B \Rightarrow B), (B \Rightarrow A) \vdash_{H_2} (\neg\neg B \Rightarrow A)$$

$B_4 = \ (A \Rightarrow \neg\neg A)$
Already proved formula: $(A \Rightarrow \neg\neg A)$

$B_5 = \ (\neg\neg B \Rightarrow \neg\neg A)$
B_3, B_4 and already proved fact: $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_2} (A \Rightarrow C)$ i.e.

$$(\neg\neg B \Rightarrow A), (A \Rightarrow \neg\neg A) \vdash_{H_2} (\neg\neg B \Rightarrow \neg\neg A)$$

$B_6 = \ ((\neg\neg B \Rightarrow \neg\neg A) \Rightarrow (\neg A \Rightarrow \neg B))$
Already proved formula: $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$ for $B = \neg B$ and $A = \neg A$

$B_7 = \ (\neg A \Rightarrow \neg B)$
B_5, B_6 and (MP) i.e.

$$\frac{(\neg\neg B \Rightarrow \neg\neg A) \ ; \ ((\neg\neg B \Rightarrow \neg\neg A) \Rightarrow (\neg A \Rightarrow \neg B))}{(\neg A \Rightarrow \neg B)}$$

$B_8 = \ (B \Rightarrow A) \vdash (\neg A \Rightarrow \neg B)$

$B_1 - B_7$

$B_9 = \ ((B \Rightarrow A) \Rightarrow (\neg B \Rightarrow \neg A))$

B_8 and Deduction Theorem

QUESTION 3 Let H' be the proof system obtained from the system H of Question 1 by extending the language to contain the negation $\neg$ and adding an additional axiom

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

(a) Is the system H' complete? Does Deduction Theorem holds for H'? Justify.

Solution Yes, it is a system $H2$ of the Book and it is proved to be complete. It is and extension of the system H for which we proved the Deduction Theorem, so it holds for it as well.

(b) Prove the following: $A \vdash_{H'} (A \Rightarrow A)$

Solution 1: Proof is as follows.

$B_1 = \ (A \Rightarrow (A \Rightarrow A))$
Axiom $A1$ for $B = A$

$B_2 = \ A$
Hypothesis

$B_3 = (A \Rightarrow A)$
B_1, B_2 and MP

Solution 2: We use Deduction Theorem.

$A \vdash_{H'} (A \Rightarrow A)$ if and only if $\vdash_{H'} (A \Rightarrow (A \Rightarrow A))$, what is true because $(A \Rightarrow (A \Rightarrow A))$ is axiom $A1$. The proof is one element sequence:

$B_1 = A \Rightarrow (A \Rightarrow A)$
Axiom $A1$ for $B = A$

(c) We know that $\vdash_{H'} (\neg A \Rightarrow (A \Rightarrow B))$. Prove, that $\neg A, A \vdash_{H'} B$.

Solution 1: We apply Deduction Theorem twice:

$\vdash_{H'} (\neg A \Rightarrow (A \Rightarrow B)$ if and only if $\neg A \vdash_{H'} (A \Rightarrow B)$ if and only if $\neg A, A \vdash_{H'} B$.

Solution 2: We construct the formal proof of $\neg A, A \vdash_{H'} B$ as follows.

$B_1 = (\neg A \Rightarrow (A \Rightarrow B))$
Assumption that $\vdash_{H'} (\neg A \Rightarrow (A \Rightarrow B)$

$B_2 = \neg A$
Hypothesis

$B_3 = A$
Hypothesis

$B_4 = (A \Rightarrow B)$
B_1, B_2 and MP

$B_5 = B$
B_3, B_4 and MP

QUESTION 4 (10pts)

Definition 0.1 *Let A be a formula and $b_1, b_2, ..., b_n$ be all propositional variables that occur in A. Let v be variable assignment $v : VAR \longrightarrow \{T, F\}$. We define, for $A, b_1, b_2, ..., b_n$ and v a corresponding formulas A', $B_1, B_2, ..., B_n$ as follows:*

$$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases}$$

$$B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$$

for $i = 1, 2, ..., n$.

Let A be a formula

$$((\neg a \Rightarrow \neg b) \Rightarrow (c \cup a))$$

and let v be such that

$$v(a) = T, \quad v(b) = F, v(c) = F.$$

Evaluate A', $B_1, ...B_n$ as defined by the above definition. Write carefully all steps. You can use shorthand notation.

Solution We use full notation.

In this case $n = 3$ and $b_1 = a,\ b_2 = b, b_3 = c$, and $v^*(A) = v^*((\neg a \Rightarrow \neg b) \Rightarrow c) =((\neg v(a) \Rightarrow \neg v(b)) \Rightarrow v(c \cup a)) = ((\neg T \Rightarrow \neg F) \Rightarrow (F \cup T) = (T \Rightarrow T) = T$.

The corresponding A', B_1, B_2, B_2 are:

$A' = A = ((\neg a \Rightarrow \neg b) \Rightarrow c)$ (as $v^*(A) = T$),

$B_1 = a$ (as $v(a) = T$),

$B_2 = \neg b$ (as $v(b) = F$).

$B_3 = \neg c$ (as $v(c) = F$).

QUESTION 5 (extra 5pts)

The lemma stated below describes a method of transforming a semantic notion of a tautology into a syntactic notion of provability. It defines, for any formula A and a variable assignment v a corresponding deducibility relation $\vdash$.

Lemma 0.1 *For any formula A and a variable assignment v, if $A^{'}$, B_1 , B_2, ..., B_n are corresponding formulas defined by* **??**, *then*

$$B_1, B_2, ..., B_n \vdash_{H2} A'. \tag{1}$$

1. Let A, v be as defined in QUESTION 4. STATE what the lemma asserts.

Solution Lemma asserts that

$$a, \neg b, \neg c \vdash_{H2} ((\neg a \Rightarrow \neg b) \Rightarrow (c \cup a))$$

2. The proof of the Lemma is by induction on the structure of A i.e. a number n of logical connectives in A; the degree of the formula A.

Write the proof of the base case, i.e the case: $n = 0$.

Solution In the case that $n = 0$ A is atomic and so consists of a single propositional variable, say a.

We have to cases to consider; $v^*(A) = T$ or $v^*(A) = F$. Clearly, if $v^*(A) = T$ then we $A' = A = a$, $B_1 = a$, and $a \vdash a$ holds by the Deduction Theorem and proven fact $\vdash_{H2}(A \vdash_{H2} A)$. I.e. $(a \vdash_{H2} a)$ holds and applying the Deduction Theorem we get $a \vdash_{H2} a$.

If $v^*(A) = F$ then we $A' = \neg A = \neg a$, $B_1 = \neg a$, and $\neg a \vdash_{H2} \neg a$ holds and by applying the the Deduction Theorem we get $\neg a \vdash \neg a$. So the lemma holds for the case $n = 0$.