## CSE371 QUIZ 4 SOLUTIONS Fall 2011

**QUESTION 1** Let **GL** be the Gentzen style proof system for classical logic (Rules includes).

**1.** Prove, by constructing a proper decomposition tree that

$$\vdash_{\mathbf{GL}}((\neg(a\cap b)\Rightarrow b)\Rightarrow(\neg b\Rightarrow(\neg a\cup\neg b))).$$

**Solution** Consider the following tree.

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow ((\neg(a\cap b)\Rightarrow b)\Rightarrow(\neg b\Rightarrow(\neg a\cup\neg b)))$$

$$\mid(\rightarrow\Rightarrow)$$

$$(\neg(a\cap b)\Rightarrow b)\longrightarrow(\neg b\Rightarrow(\neg a\cup\neg b))$$

$$\mid(\rightarrow\Rightarrow)$$

$$\neg b,(\neg(a\cap b)\Rightarrow b)\longrightarrow(\neg a\cup\neg b)$$

$$\mid(\rightarrow\cup)$$

$$\neg b,(\neg(a\cap b)\Rightarrow b)\longrightarrow\neg a,\neg b$$

$$\mid(\rightarrow\neg)$$

$$b,\neg b,(\neg(a\cap b)\Rightarrow b)\longrightarrow\neg a$$

$$\mid(\rightarrow\neg)$$

$$b,a,\neg b,(\neg(a\cap b)\Rightarrow b)\longrightarrow$$

$$\mid(\neg\rightarrow)$$

$$b,a,(\neg(a\cap b)\Rightarrow b)\longrightarrow b$$

$$\bigwedge(\Rightarrow\rightarrow$$

| | |
|---|---|
| $b,a\longrightarrow\neg(a\cap b),b$ | $b,a,b\longrightarrow b$ |
| $\mid(\rightarrow\neg)$ | *axiom* |
| $b,a,(a\cap b)\longrightarrow b$ | |
| $\mid(\cap\rightarrow)$ | |
| $b,a,a,b\longrightarrow b$ | |
| *axiom* | |

All leaves of the decomposition tree are axioms, hence the proof has been found.

**2.** Show that $\nvdash_{\mathbf{GL}}((a\Rightarrow b)\Rightarrow(\neg b\Rightarrow a))$.

**Solution** Consider the following trees.

$$\mathbf{T}_1$$

$$\longrightarrow ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\to\Rightarrow)$$

$$(a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\to\Rightarrow)$$

$$\neg b, (a \Rightarrow b) \longrightarrow a$$

$$\mid (\to\Rightarrow)$$

$$(a \Rightarrow b) \longrightarrow b, a$$

$$\bigwedge (\Rightarrow\to$$

$$\longrightarrow a, b, a \qquad\qquad b \longrightarrow b, a$$

$$non - axiom \qquad\qquad axiom$$

The tree has a non- axiom leaf, so it is not a proof. To prove that a proof does not exist in GL we must consider all possible decomposition trees; in this case there is only one more (the second choice in the second step).

$$\mathbf{T}_1$$

$$\longrightarrow ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$$

$$\mid (\to\Rightarrow)$$

$$(a \Rightarrow b) \longrightarrow (\neg b \Rightarrow a)$$

$$\bigwedge (\Rightarrow\to$$

$$\longrightarrow (\neg b \Rightarrow a), a \qquad\qquad b \longrightarrow (\neg b \Rightarrow a)$$

$$\mid (\rightarrow\Rightarrow) \qquad\qquad \mid (\rightarrow\Rightarrow)$$

$$\neg b \longrightarrow a, a \qquad\qquad b, \neg b \longrightarrow a$$

$$\mid (\neg \rightarrow) \qquad\qquad \mid (\neg \rightarrow)$$

$$\longrightarrow b, a, a \qquad\qquad b \longrightarrow b, a$$

$$non-axiom \qquad\qquad axiom$$

These are all possible decomposition trees and none proof; hence the proof does not exist.

**3.** Use the above (i.e. the decomposition tree) and strong soundness to prove that $\not\models ((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$

**Solution** The Tree 2 has a non-axiom leaf: $\longrightarrow b, a, a$. The counter-model determine by this leaf is any $v$ such that $v(a) = v(b) = F$. By strong soundness this also is a counter-model for $((a \Rightarrow b) \Rightarrow (\neg b \Rightarrow a))$.

**QUESTION 2** We know that a classical tautology $(\neg(A \cap B) \Rightarrow (\neg A \cup \neg B))$ is NOT Intuitionistic tautology.

Show, that the formula $A = \neg\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$ is PROVABLE in the Gentzen system **LI** for Intuitionistic Logic (rules included), i.e. that

$$\vdash_{\mathbf{LI}} \neg\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

**Solution**

$$\mathbf{T}_{\rightarrow A}$$

$$\longrightarrow \neg\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\rightarrow \neg)$$

$$\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow$$

$$\mid (contr \rightarrow)$$

$$\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)), \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$\mid (\rightarrow\Rightarrow)$$

$$\neg(a \cap b), \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow (\neg a \cup \neg b)$$

$$\mid (\rightarrow \cup)_1$$

$$\neg(a \cap b), \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow \neg a$$

$$\mid (\rightarrow \neg)$$

$$a, \neg(a \cap b), \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow$$

$$| \ (exch \longrightarrow)$$

$$\neg(a \cap b), a, \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow$$

$$| \ (\neg \longrightarrow)$$

$$a, \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow (a \cap b)$$

$$\bigwedge(\longrightarrow \cap$$

$$a, \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow a) \qquad\qquad a, \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow b$$

$$axiom \qquad\qquad | \ (\longrightarrow weak)$$

$$a, \neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)) \longrightarrow$$

$$| \ (exch \longrightarrow)$$

$$\neg(\neg(a \cap b) \Rightarrow (\neg a \cup \neg b)), a \longrightarrow$$

$$| \ (\neg \longrightarrow)$$

$$a \longrightarrow (\neg(a \cap b) \Rightarrow (\neg a \cup \neg b))$$

$$| \ (\longrightarrow \Rightarrow)$$

$$\neg(a \cap b), a \longrightarrow (\neg a \cup \neg b)$$

$$| \ (\longrightarrow \cup)_2$$

$$\neg(a \cap b), a \longrightarrow \neg b$$

$$| \ (\longrightarrow \neg)$$

$$b, \neg(a \cap b), a \longrightarrow$$

$$| \ (exch \longrightarrow)$$

$$\neg(a \cap b), b, a \longrightarrow$$

$$| \ (\neg \longrightarrow)$$

$$b, a \longrightarrow (a \cap b)$$

$$\bigwedge(\longrightarrow \cap$$

$$b, a \longrightarrow a \qquad\qquad b, a \longrightarrow b$$

$$axiom \qquad\qquad axiom$$

All leaves are axioms, the tree is a proof of $A$ in **LI**.

**QUESTION 3** Show, that (use heuristic and observations)

$$\nvdash_{\mathbf{LI}} (\neg(A \cap B) \Rightarrow (\neg A \cup \neg B))$$

**Solution** Use the Heuristic (finite number of trees!) given in the chapter!

**SHORT QUESTIONS** (Extra 5 points)

**Circle proper answer. Write one sentence justification**

1. For any predicates A(x), B(x),
$\neg\forall x(A(x) \cap B(x)) \equiv (\exists x \neg A(x) \cup \exists x \neg B(x))$.
JUSTIFY: $\neg\forall x(A(x) \cap B(x)) \equiv \exists x(\neg A(x) \cup \neg B(x)) \equiv (\exists x \neg A(x) \cup \exists x \neg B(x))(distr(\exists, \cup))$ **y**

2. $\forall x(A(x) \cap B(x)) \equiv (\forall x A(x) \cap \forall x B(x))$
JUSTIFY: Use Truth sets and fact that for any non empty set $X$ and any sets $A, B \subseteq X$ we have that $A \cap B = X$ iff $A = X$ and $B - X$. **y**

3. $\exists x(A(x) \cup B(x)) \equiv (\exists x A(x) \cup \exists x B(x))$
JUSTIFY: Use Truth sets and fact that for any non empty set $X$ and any sets $A, B \subseteq X$ we have that $A \cup B = X$ iff $A = X$ or $B - X$. **y**

4. $\exists x(x < 1) \cup 2 + 2 = 4$ is a true statement in a set of natural numbers numbers.
JUSTIFY: $T \cup F = T$ **y**

5. $\neg\exists n \exists x(x < \frac{1+n}{n-1}) \equiv \forall n \exists x(x \geq \frac{1+n}{n-1}))$
JUSTIFY: $\neg\exists n \exists x(x < \frac{1+n}{n-1}) \equiv \forall n \forall x \neg(x < \frac{1+n}{n-1}) \equiv \forall n \forall x(x \geq \frac{1+n}{n-1})$ **n**

6. $\exists x A(x) \Rightarrow \forall x A(x)$ is a predicate tautology.
JUSTIFY: Take: $X = R, A(x) = x > 0$. We get $\exists x x > 0 = T$ and $\forall x x > 0 = F$. **n**

7. $\neg\forall x(A(x) \cap B(x)) \equiv (\neg\forall x A(x) \cup \exists x \neg B(x))$.
JUSTIFY: $\neg\forall x(A(x) \cap B(x)) \equiv \exists x \neg(A(x) \cap B(x)) \equiv \exists x(\neg A(x) \cup \neg B(x)) \equiv (\exists x \neg A(x) \cup \exists x \neg B(x)) \equiv (\neg\forall x A(x) \cup \exists x \neg B(x))$. **y**

8. The formula $\forall x(C(x) \cap F(x))$ represents sentence: *All birds can fly* in in the domain $X \neq \emptyset$.
JUSTIFY: $X \neq |emptyset$, $B(x)$ – x is a bird, $F(x)$ – x flies, $\forall B(X) F(x) \equiv |forall x(B(x) \Rightarrow F(x))$ **n**

9. For any predicates A(x), B, (this means that B does not contain the variable x) the formula $\forall x(A(x) \Rightarrow B) \Rightarrow (\exists x A(x) \Rightarrow B)$ is a predicate tautology.
JUSTIFY: $\forall x(A(x) \Rightarrow B) \equiv \forall x(\neg A(x) \cup B) \equiv (\forall x \neg A(x) \cup B) \equiv (\neg\exists x A(x) \cup B) \equiv (\exists x A(x) \Rightarrow B)$ **y**